ACADEMIC TASK FORCE

MATHEMATICS METHODS

Year 11 ATAR COURSE

Units 1 and 2

WACE Revision Series

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The Mathematics Methods Revision Series Units 1 & 2 provides a comprehensive set of revision/review questions for the West Australian Mathematics Methods Units 1 & 2. It is accompanied by a set of fully worked solutions, which doubles as a set of model answers.

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Mathematics Methods Revision Series Units 1 & 2

Contents

Mathematics Methods Revision Series
Units 1 & 2

- The Mathematics Methods Revision Series Units 1 & 2 provides a comprehensive set of revision/review questions for the new year 11 Mathematics Methods Units 1 & 2 course.

- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.

- This book exposes students to questions and problems at test/examination level.

- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.

- It is accompanied by a set of fully worked solutions with which students can measure their solutions. These solutions are often not the only solutions but they provide a model for students to work with. Students, interrogate your solutions to understand your errors and your successes. It may sometimes to possible to achieve a correct numerical answer with faulty reasoning!

- Do not memorise solutions. Understand the techniques and processes used in relation to the questions asked.

Notes

Linear Functions

- Equation of a line with gradient m and vertical intercept c is $y = mx + c$.
- Equation of a line with gradient m and passing through the point (h, k) is $y - k = m(x - h)$.
- Two lines with gradients m_1 and m_2 respectively are
  (i) parallel if $m_1 = m_2$.
  (ii) perpendicular if $m_1 \times m_2 = -1$.

Quadratic Functions

- For the parabola $y = a(x-h)^2 + k$: turning point is (h, k).
- For the parabola $y = ax^2 + bx + c$: line of symmetry is $x = -\dfrac{b}{2a}$. (this is the x-coordinate of the turning point)
- For the parabola $y = a(x-p)(x-q)$: line of symmetry is $x = \dfrac{p+q}{2}$. (this is the x-coordinate of the turning point)
- Parabola has a minimum point if the coefficient of the x^2 term is positive, maximum otherwise.
- For the equation $ax^2 + bx + c = 0$ roots are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  Roots are real and different if $b^2 > 4ac$.
  Roots are real and repeated if $b^2 = 4ac$.
  Roots are complex if $b^2 < 4ac$.

Cubics

- For the cubic $y = ax^3 + bx^2 + cx + d$:

Factors of $ax^3 + bx^2 + cx + d$	No. of roots
3 distinct linear factors	3
3 linear factors with 2 the same	2
all 3 linear factors the same	1
1 linear and 1 non-reducible quadratic	1

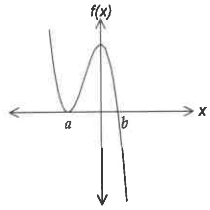
Rectangular Hyperbola

$y = \dfrac{k}{x-a}$ has:

- a horizontal asymptote with equation $y = 0$.
- a vertical asymptote with equation $x = a$.

Polynomials

- To find the equation of the given curve: use roots with a multiplier k.
  $y = k(x-a)^2(x-b)$
  Root is repeated when the curve bounces off the x-axis at that root.



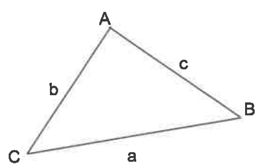
Functions

- A relation r between sets X and Y is a rule that associates (maps) elements in set X with elements in set Y.
- A function f between sets X and Y is a **rule** that associates **each** element in set X with a **unique** element in set Y.
- A function f has either a 1 to 1 rule or a 1 to many rule.
- The graph of function f passes the "vertical line" test'.
-

Functions	Domain	Range
$y = \sqrt{x-a} + b$	$x \geq a$	$y \geq b$
$y = a^x + b$	**R**	$y > b$
$y = \dfrac{k}{x-a}$	$x \neq a$	$y \neq 0$

- Circles are relations with equations that can be written in the form:
  $(x-a)^2 + (y-b)^2 = r^2$.
  - centre (a, b)
  - radius r.
- For $y = -k f(-ax + b) + m$:
  1. Translate b units left along the x-axis — **T**
  2. Dilate along the x-axis by factor $1/a$ — **D**
  3. Reflect about the y-axis — **R**
  4. Reflect about the x-axis — **R**
  5. Dilate along the y-axis by factor k — **D**
  6. Translate m units up along the y-axis — **T**

Non-Right Triangles

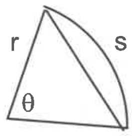


- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$
- $a^2 = b^2 + c^2 - 2bc \cos A$
- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- Area of Triangle $= \frac{1}{2} ab \sin C$

Arcs and Sectors

Angle θ is in radians

- Arc length $s = r\theta$
- Area of sector = $\frac{1}{2}r^2\theta$
- Area of segment = $\frac{1}{2}r^2(\theta - \sin\theta)$



Exact Values

θ°	θ rad	sin θ	cos θ	tan θ
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	∞

Trigonometric Identities

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

- Trig Graphs

	$y = a\sin(bx + c) + d$ $y = a\cos(bx + c) + d$
Mean Line	$y = d$
Amplitude	$\lvert a\rvert$
Min./Max. y	Min: $d - \lvert a\rvert$, Max: $d + \lvert a\rvert$
Period	$360°/b$ or $2\pi/b$
Phase shift	*Shifted c/b degrees/radians to the left*

	$y = a\tan(bx + c) + d$
Mean Line	$y = d$
Period	$180°/b$ or π/b
Phase shift	*Shifted c/b degrees/radians to the left*

Set Notation

Symbol	Meaning
$\in$	is an element of
$\subset$	is a subset of
$\cap$	intersection
$\cup$	union
$n(A)$ or $\lvert A\rvert$	No. of elements in set **A**
A′ or $\overline{A}$	Complement of A

Combinations

- $^nC_r \equiv \binom{n}{r} \equiv \dfrac{n!}{r!\,(n-r)!}$

$$= \frac{\overbrace{n\times(n-1)\times(n-2)\times\ldots\ldots\times(n-r+1)}^{r\ \text{terms}}}{r!}.$$

- $^nC_r = {}^nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$.
- $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = \binom{n}{n-1} = n$.
- r items can be chosen from n items all different:
  - without replacement in ^nC_r ways
  - with replacement in n^r ways.
- $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2$

$$\ldots + \binom{n}{k}x^{n-k}y^k + \ldots$$

$$+ \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Probability

- $0 \le P(A) \le 1$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A) \times P(B|A)$
- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
- $P(\overline{A}) = 1 - P(A)$
- Two events A and B are mutually exclusive if $P(A \cap B) = 0$.
- To show that A and B are mutually exclusive, show that $P(A \cap B) = 0$.
- Two events A and B are independent if $P(A|B) = P(A)$ or $P(B) = P(B|A)$.
- To show that A and B are independent:
  - show that $P(A|B) = P(A)$ or $P(B) = P(B|A)$
  - or show that $P(A \cap B) = P(A) \times P(B)$.

Indices

- $a^x \times a^y = a^{x+y}$ $\qquad \dfrac{a^x}{a^y} = a^{x-y}$

$(a^x)^y = a^{xy}$ $\qquad a^0 = 1$

$\dfrac{1}{a^x} = a^{-x}$ $\qquad \sqrt[n]{a} = a^{\frac{1}{n}}$

Arithmetic Progression

- General Rule for the nth term:
  $T_n = a + (n-1)d$
- Recursive equation:
  $T_{n+1} = T_n + d \qquad T_1 = a$
- Sum of first n terms:
  $$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$$

Geometric Progression

- General Rule for nth term: $T_n = a \times r^{n-1}$
- Recursive equation:
  $T_{n+1} = T_n \times r \qquad T_1 = a$
- Sum of first n terms:
  $$S_n = \frac{a(1-r^n)}{1-r} \quad \text{where } r \neq 1$$
- Sum to infinity for $-1 < r < 1$:
  $$S_\infty = \frac{a}{1-r}$$

Exponential Growth and Decay

- $P(t+1) = P(t) \times r$ where $P(0)$ = initial value
- $P(t) = P(0)r^t$

Differentiation

- $f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$
- $y = a\,x^n \Rightarrow y' = n \times a\,x^{n-1}$

Rate of change

- The instantaneous rate of change of Q at time $t = a$ is $Q'(a)$.
- The average rate of change between $t = a$ and $t = b$ is $\frac{Q(b)-Q(a)}{b-a}$

Features of graphs

$y = f(x)$	$y = f'(x)$
max point	x-intercept (crosses x-axis from above to below)
min point	x-intercept (crosses x-axis from below to above)
inflection point	turning point

Stationary & Inflection Points

- For max point at $x = a$: $y' = 0$,

x	a^-	a	a^+
y'	+	0	−

- For min point at $x = a$: $y' = 0$,

x	a^-	a	a^+
y'	−	0	+

- For horizontal inflection point at $x = a$:

x	a^-	a	a^+
y'	±	0	±

Integration

- $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C \qquad [n \neq -1]$

Rectilinear Motion

- Displacement at time t, $x = \int v\,dt$

  Velocity $v = \frac{dx}{dt} = \int a\,dt$

  Acceleration $a = \frac{dv}{dt}$
- Body changes direction when $v = 0$.
- Body returns to origin when $x = 0$.

01 Lines

Calculator Free

1. [5 marks: 1, 2, 2]

   A line passes through the points (1, 2) and (5, 22).

   (a) Find the gradient of this line.

   (b) Find the equation of this line.

   (c) Is (3, 25) on this line? Justify your answer.

---

2. [3 marks]

   The points (–2, 5), (3, k) and (5, 12) are collinear. Find the value(s) of k.

---

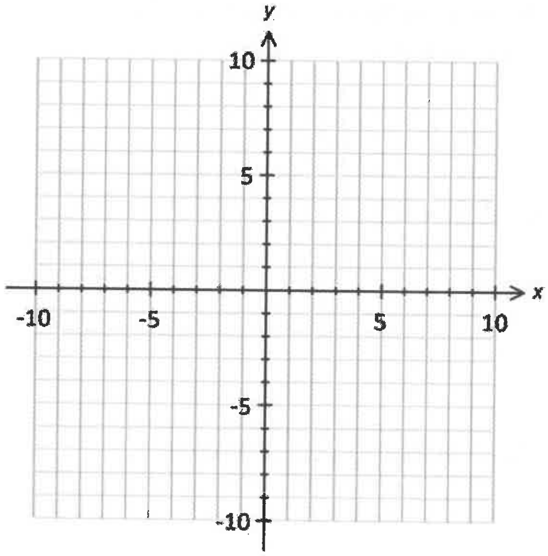
3. [3 marks]

   The line $ax + by = 18$ passes through the point (1, –4) and has a gradient of 2. Find a and b.

Calculator Free

4. [2 marks]

Sketch the line with equation $3x + 5y = 15$.
Indicate clearly on your sketch the intercepts of this line.



5. [6 marks: 3, 3]

Find the equation of the line passing through the point with coordinates (10, 3):

(a) and parallel to the line with equation $4x + 5y = 20$.

(b) and perpendicular to the line with equation $2x + 3y = 12$.

Calculator Free

6. [7 marks: 3, 2, 2]

The lines $2x + 3y = 12$ and $4x + 5y = 20$ meet at the point P.

(a) Find the coordinates of P.

(b) Find the equation of the line through P and parallel to the line with equation $2x + y = 10$.

(c) Find the equation of the line through P and perpendicular to the line with equation $2x + y = 10$.

---

7. [3 marks: 2, 1]

Consider the line $2x + by = c$ where c is a constant.

(a) Find b if this line has gradient -4.

(b) Find c if this line has an x–intercept of 6.

Calculator Free

8. [6 marks: 2, 2, 2]

Suggest one possible equation each for the lines L1 and L2 if:

(a) L1 and L2 are each parallel to $x + 2y = 0$.

(b) L1 and L2 meet at the point with coordinates (0, 4) and are perpendicular to each other.

(c) L1 and L2 do not intersect.

---

9. [5 marks: 3, 2]

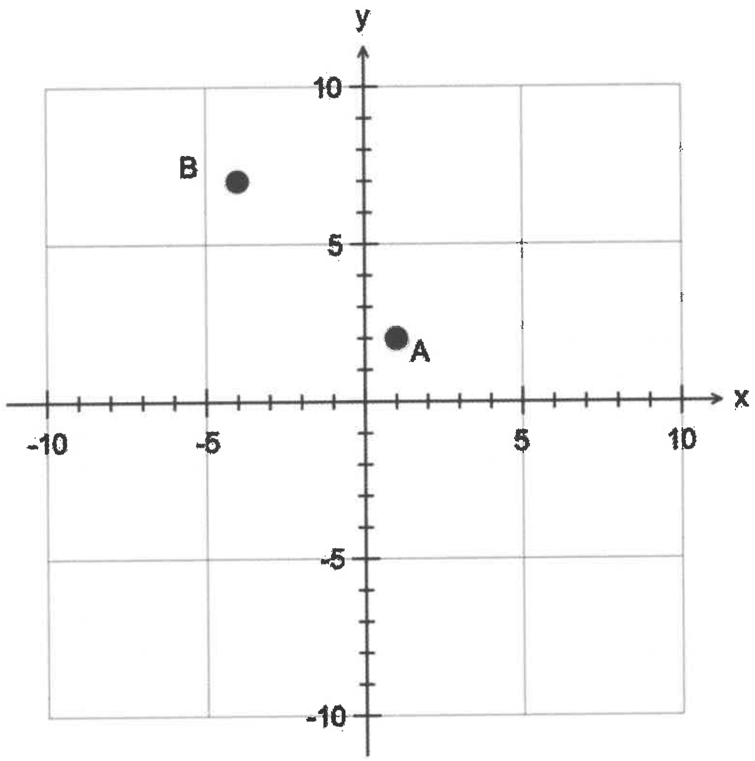
The line with equation $7x + 5y = 70$ intersects the x-axis and y-axis at A and B respectively.

(a) Find the coordinates of the mid-point of AB.

(b) Find the distance between A and B.

Calculator Free

10. [9 marks: 4, 5]

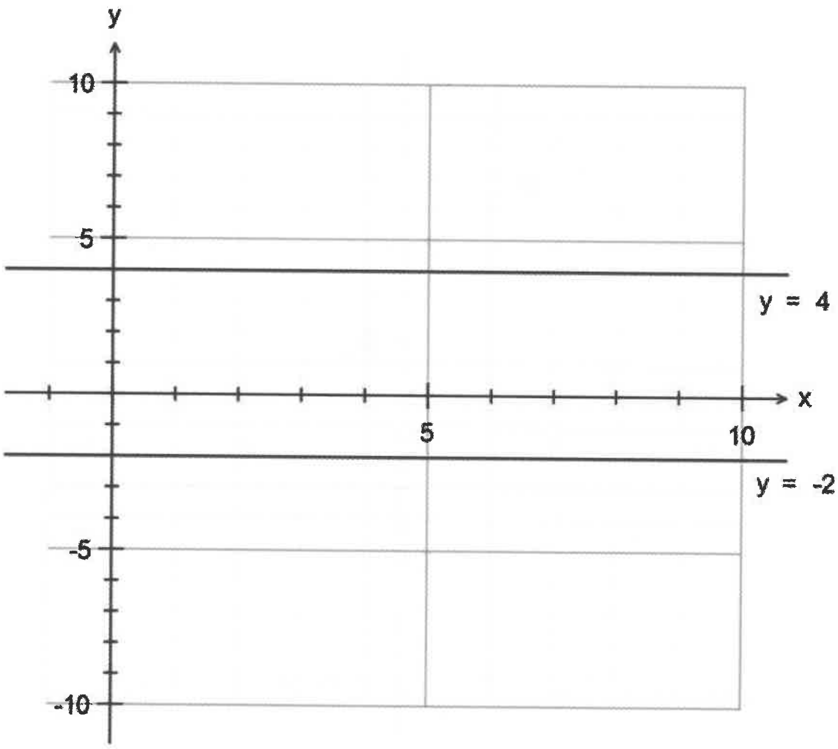


(a) On the axes provided above, sketch the line L1 which passes through the point A (1, 2) with gradient $\frac{1}{2}$. Also sketch the line L2 which passes through the point B (–4, 7) and perpendicular L1.

(b) Use your graph to find the coordinates of C the point of intersection between the lines in (a) and (b). Hence, find the area of Δ ABC.
[Hint: $\sqrt{5} \times \sqrt{45} = 15$]

Calculator Free

11. [7 marks: 3, 2, 2]



The diagram above shows the lines with equations $y = -2$ and $y = 4$.

(a) On the diagram above draw the line L1 passing through the point (0, –4) with gradient 2. Also, draw the line L2 parallel to L1 but passing through the point (5, 0).

(b) The line L1 meets the lines $y = -2$ and $y = 4$ at P and Q respectively. The line L2 meets the lines $y = -2$ and $y = 4$ at S and R respectively.

(i) On the diagram above, clearly mark the points P, Q R and S. Find the area of PQRS.

(ii) Find the perimeter of PQRS.

Calculator Assumed

12. [10 marks: 2, 1, 3, 4]

Bill, a plumber charges a call-out fee of $100 plus $80 per half hour or part thereof. Ian, another plumber does not charge a call-out fee but charges $180 per hour or part thereof.

(a) How much will Bill charge for a job that is estimated to take exactly 4 hours?

(b) How much will Ian charge for a job that is estimated to take exactly 4 hours?

(c) Determine, which plumber will be cheaper to employ if a job is estimated to take 3 hours and 20 minutes. Justify your answer.

(d) Under what conditions will it be cheaper to employ Bill?
Justify your answer.

02 Quadratics

Calculator Free

1. [8 marks: 3, 2, 3]

A parabola has equation $y = (x - 2)(5 - x)$.

(a) Find the coordinates of the x and y intercepts of the parabola .

(b) Find the equation of the line of symmetry.

(c) Find the coordinates of the turning point of the parabola and state the nature of the turning point.

---

2. [6 marks: 3, 3]

A parabola has equation $y = 10 - 6x - 3x^2$.

(a) Find the coordinates of the turning point and state its nature.

(b) Find the exact x-intercepts.

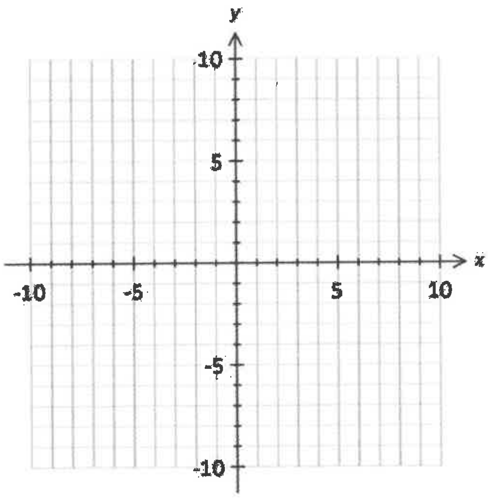
Calculator Free

3. [6 marks: 3, 3]

A parabola has equation $y = (x-2)^2 - 5$.

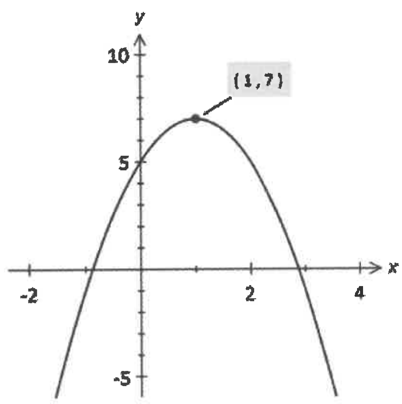
(a) Find the exact coordinates of the x-intercepts.

(b) Sketch this parabola. Indicate clearly the coordinates of the intercepts and the turning point.



4. [3 marks]

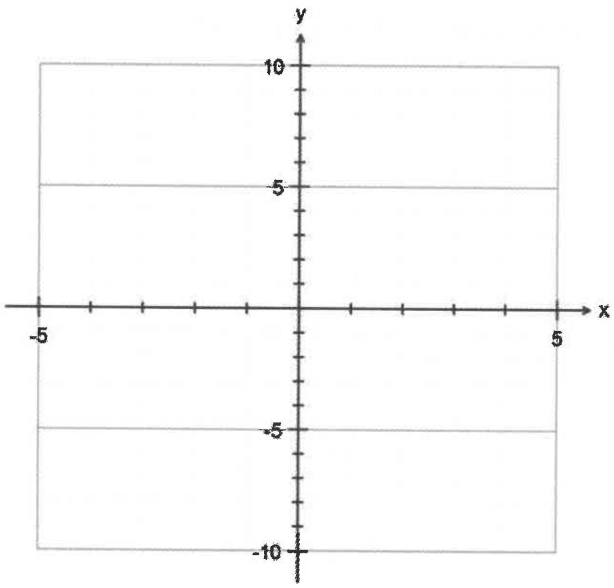
Find the equation of the parabola shown in the accompanying diagram.



Calculator Free

5. [4 marks]

Sketch the parabola with equation $y = 2x^2 + x - 6$. Indicate clearly the intercepts and the turning point.



6. [4 marks]

A parabola has equation $y = -x^2 + 4x + 5$. Rewrite the equation of this parabola in the form $y = k\,(x - a)^2 + b$, giving the values of a, b and k.

7. [3 marks]

A parabola has equation $y = -x^2 + bx + c$. Find the values of b and c, if the parabola has a turning point at $(-1, 4)$ and an intercept at $(0, 3)$.

Calculator Free

8. [4 marks]

A parabola has equation $y = k(x - a)(x - b)$ where k, a and b are constants with $a < b$. Find a, b and k if the parabola has an x-intercept at $(-3, 0)$, a turning point at $(1, 32)$ and a y-intercept at $(0, 30)$.

9. [6 marks: 1, 1, 2, 2]

Consider the parabola with equation $y = f(x) = (x - 2)(x + a)$ where a is a constant.

(a) Find a if the parabola has exactly one root.

(b) Find a if $f(2) = f(4) = 0$.

(c) Find a if $f(0) = 10$.

(d) Find a if the parabola has a turning point at $x = 3$.

Calculator Free

10. [12 marks: 2, 3, 3, 2, 2]

A parabola has equation $y = f(x)$ where $f(x) = k(x + a)^2 + 16$ where a is a constant.

(a) Find a and k if the parabola has a turning point at (1, 16).

(b) Find a and k if the parabola has a turning point at (–2, 16) and $f(0) = -4$.

(c) Find a and k if $f(3) = f(-5) = 0$.

(d) Find k if the parabola has no roots.

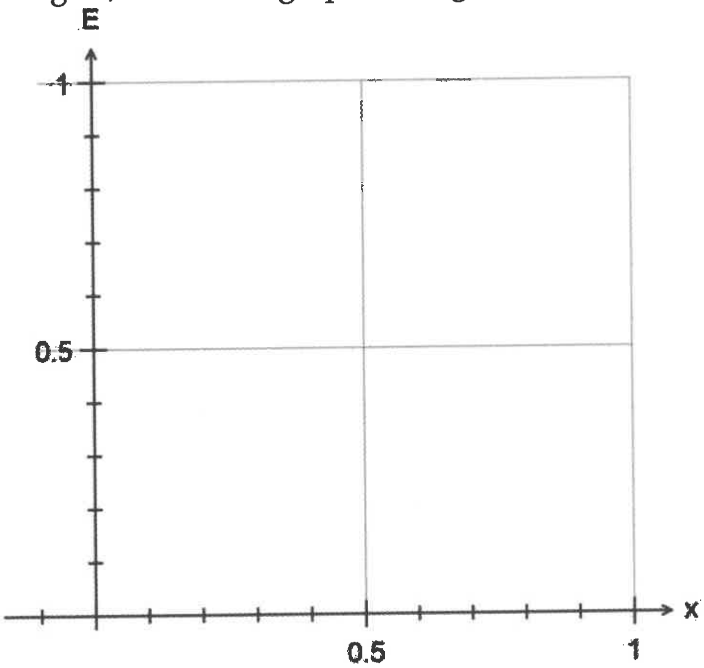
(e) Explain clearly why the parabola cannot have exactly one root.

Calculator Assumed

11. [6 marks: 2, 1, 1, 2]

The efficiency rating, E, of a spark plug when the gap is set at x mm is given by $E = 3x(1 - x)$.

(a) In the given grid, sketch the graph of E against x for $0 \le x \le 1$.



(b) What values of x would give an efficiency rating of zero?

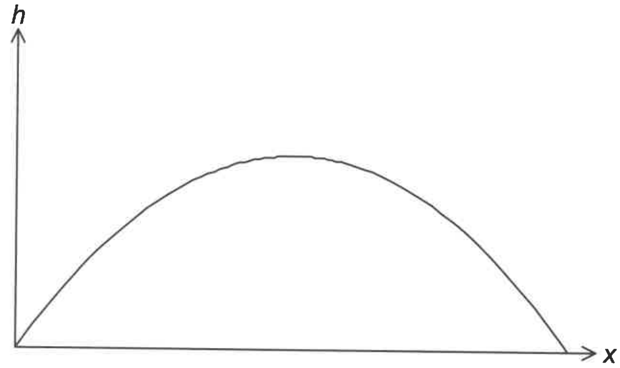
(c) What is the value of the maximum efficiency rating?

(d) Find the values of x between which the efficiency rating is 0.6 or more.

Calculator Assumed

12. [6 marks: 1, 1, 2, 2]

The height (h metres) of a soccer ball in flight is given by $h = -0.008x(x-40)$ for $x \geq 0$, where x (metres) is the horizontal distance travelled from the point where the ball was kicked. Assume that the ball travels in a vertical plane.



Use an appropriate method, stating clearly the method you have used, (either using algebra or using your CAS/graphic calculator) to find:

(a) the maximum height reached by the ball

(b) the horizontal distance travelled by the ball if it was not intercepted during its flight

(c) the horizontal distance travelled by the ball if it was intercepted at a height of 2 metres.

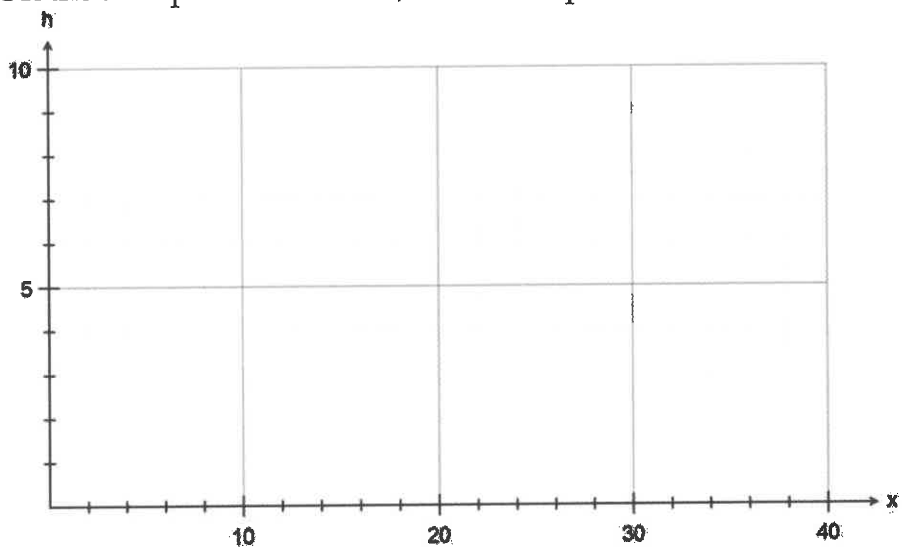
(d) x when the ball was more than 1 metre above the ground.

Calculator Assumed

13. [7 marks: 3, 1, 1, 2]

The height (h metres) of a cricket ball in flight is given by $h = 1 + \frac{3}{5}x - \frac{1}{50}x^2$ for $x \geq 0$, where x (metres) is the horizontal distance travelled from the point where the ball was struck by a bat. Assume that the ball travels in a vertical plane.

(a) On the axes provided below, sketch the path of the cricket ball.



Use an appropriate method, showing clearly the method you have used, (either using algebra or using your CAS/graphic calculator) to find:

(b) the height at which the ball was struck.

(c) the maximum height reached by the ball.

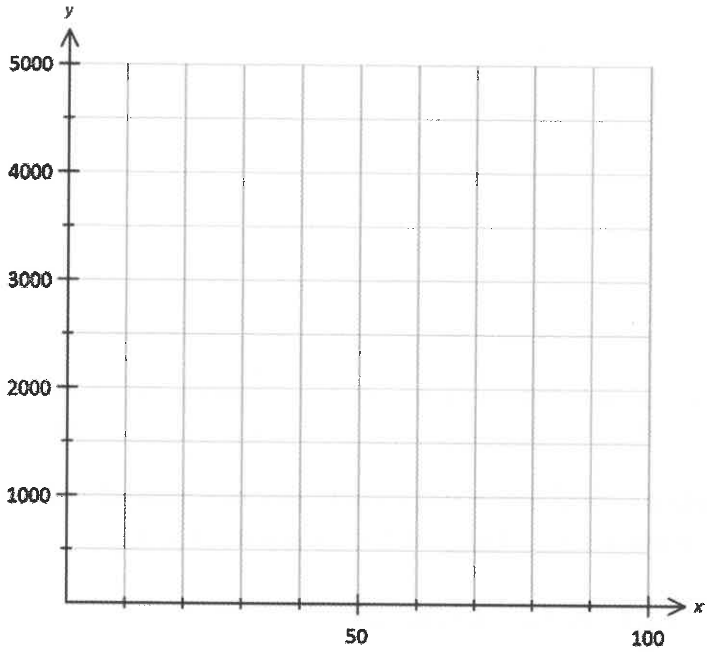
(d) the horizontal distance travelled by the ball if it was caught when it was 2 m above the ground.

Calculator Assumed

14. [7 marks: 2, 1, 2, 2]

Gemma owns a hobby farm and needs to create a fenced up area for her sheep using the back wall of her shed as one of the sides of the fenced up area. She has 200 metres of fencing available. From what she could recall from her mathematics class when she was a student, to maximise the fenced up area, she would need to maximise the function $A(x) = x(200 - 2x)$ where x is the width of the fenced up area.

(a) On the axes provided below sketch $A(x) = x(200 - 2x)$.



(b) Find the coordinates of the turning point of function A(x).

(c) Find the maximum possible area that can be fenced and the dimensions of that fenced up area.

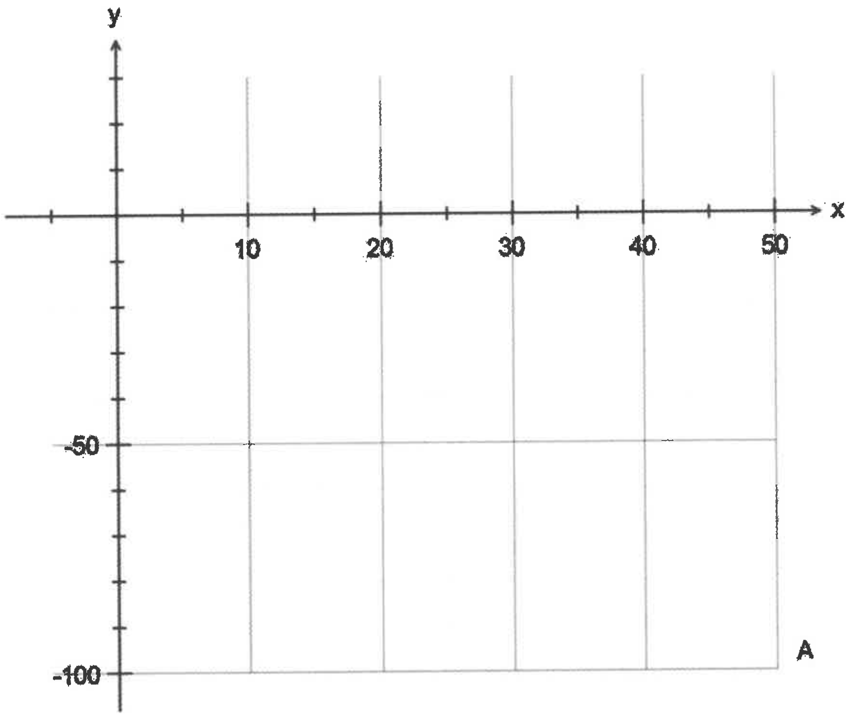
(d) Find the possible dimensions of the fenced up area if its area is 3200 m^2.

Calculator Assumed

15. [8 marks: 3, 1, 1, 3]

A ball is thrown off the top of a cliff, 100 m above sea level. Taking the point of projection O as the origin of the coordinate axes, the path taken by the ball is given as $y = 0.1x(30 - x)$. The ball hits the surface of the sea at A.

(a) On the axes provided below, sketch the path of the ball. Mark the point A on your sketch.



(b) Write the equation for the surface of the sea.

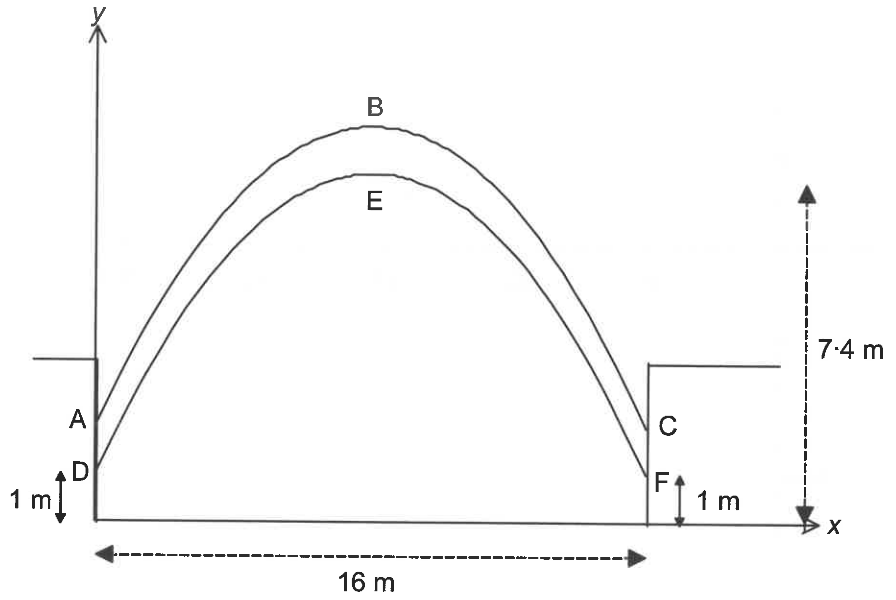
(c) Find the distance from A to B, the base of the cliff .

(d) Find the horizontal distance from O when the ball is 110m above sea level.

Calculator Assumed

16. [9 marks: 3, 1, 1, 1, 3]

The diagram below shows the cross section of an arch. ABC and DEF are the top and bottom edges of the arch respectively. Each of these edges is approximately parabolic in shape. The edges ABC and DEF are "parallel" with ABC positioned 1 metre above DEF. D and F are each 1 metre above the road level. The road level is modelled by the x-axis. The vertical line through D and A is modelled by the y-axis.



(a) Find the equation of the bottom edge of the arch (DEF).

(b) Find the equation of the top edge of the arch (ABC).

Calculator Assumed

16. (c) Find the coordinates of B, the highest point on the arch.

(d) What is the clearance of the arch (above road level) at a point which is horizontally 5 m from D?

(e) At what horizontal distance from D is the clearance of the arch 5 m above the road level?

03 Cubics

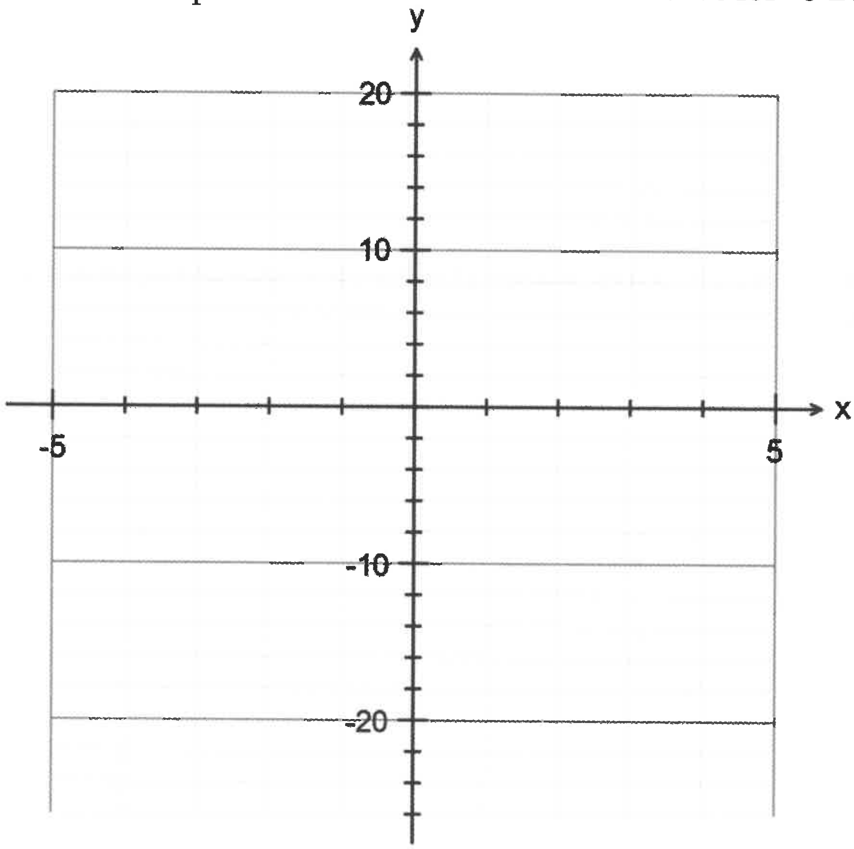
Calculator Free

1. [4 marks: 1, 3]

A curve has equation $y = -(x-1)(x+2)(3-x)$.

(a) Find y when $x = -3$ and $x = 4$.

(b) Find the intercepts of this curve and sketch this curve for $-3 \le x \le 4$.



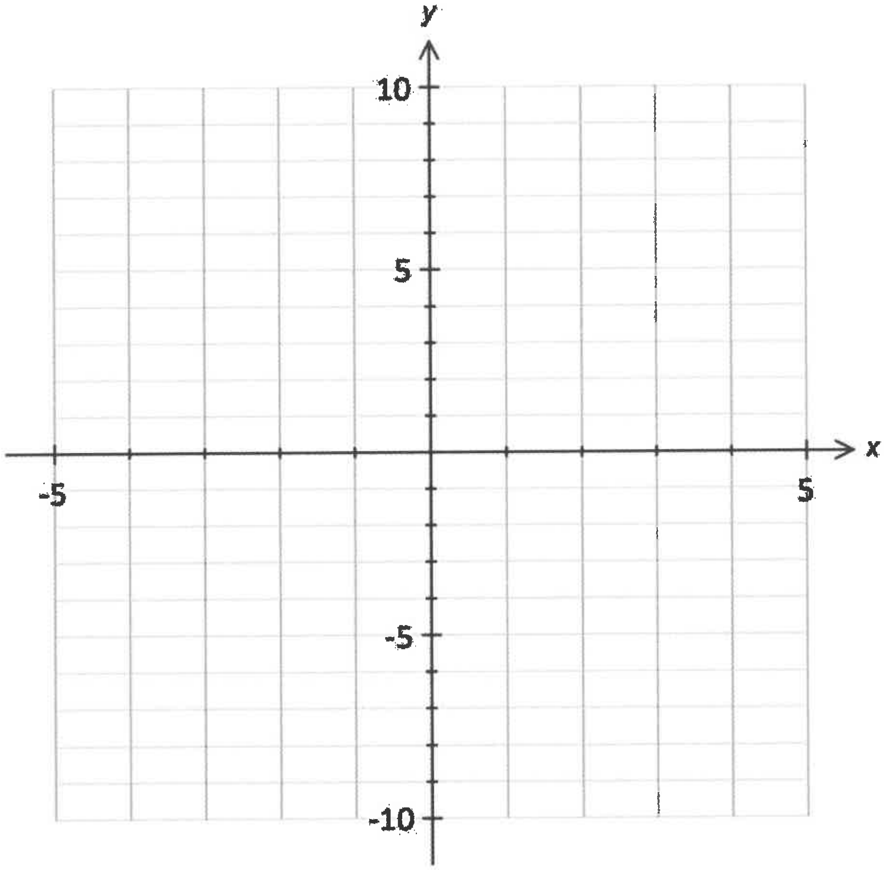
Calculator Free

2. [7 marks: 4, 3]

A curve has equation $y = 2x^3 - x^2 - 2x + 1$.

(a) Find the coordinates of the x-intercepts of this curve.

(b) Sketch this curve for $-1.5 \le x \le 2$.

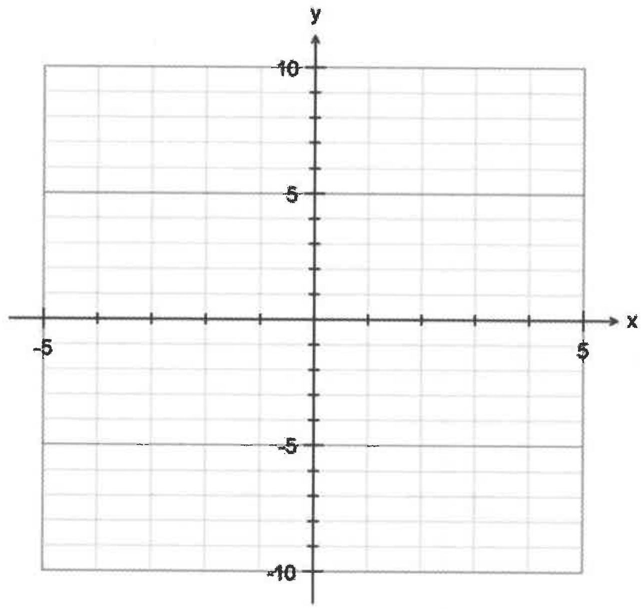


Calculator Free

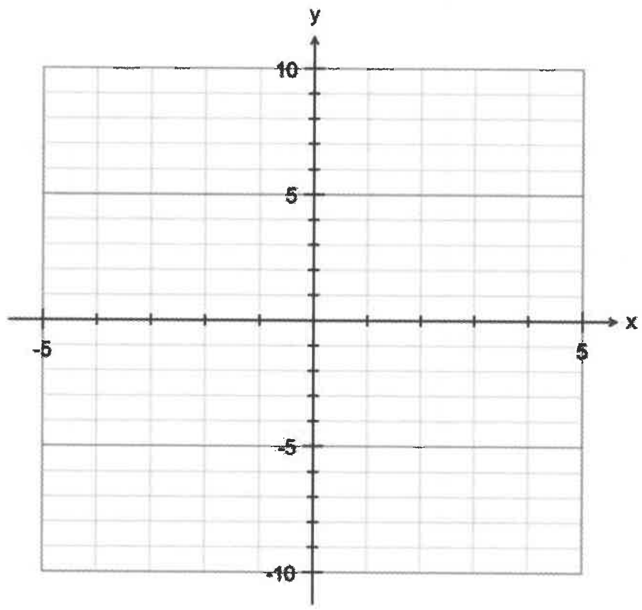
3. [6 marks: 3, 3]

Give a possible sketch for each of the following cubic curves:

(a) A cubic with roots $x = -2, 4$ and y-intercept $(0, -5)$.



(b) A cubic with root $x = 1$ and y-intercept $(0, 7)$.

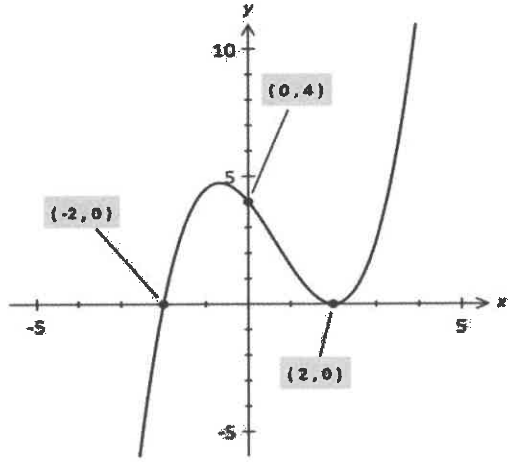


Calculator Free

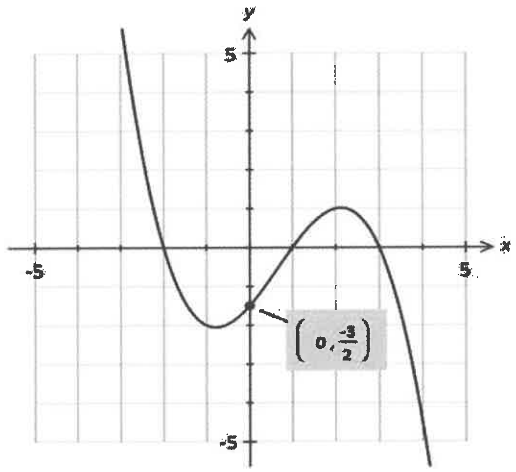
4. [9 marks: 3, 3, 3]

Determine the equations of each of the following cubic curves.

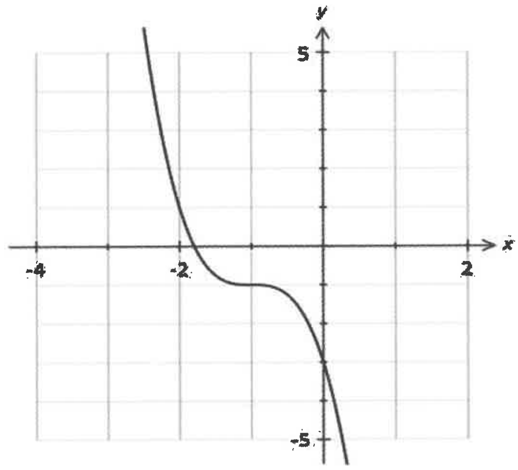
(a)



(b)



(c)



Calculator Free

5. [7 marks: 1, 2, 2, 2]

Equations of cubic curves can be written in the form $y = k(x - a)(x - b)(x - c)$ or $y = k(x - a)(x - b)^2$ or $y = ax^3 + bx^2 + cx + d$. Find a possible equation of a cubic curve if this curve has:

(a) exactly three roots $x = 1, 2, -1$.

(b) exactly three roots $x = 1, 2, -1$ and y-intercept at $(0, -6)$

(c) exactly two roots $x = -1, 1$ and y-intercept at $(0, -6)$.

(d) has exactly one root $x = 1$ and y-intercept $(0, 2)$

---

6. [6 marks: 1, 1, 2, 1, 1]

Consider the cubic curves :

I $\quad y = (x - 1)(x + 2)^2$

II $\quad y = (x + 1)(x^2 - 1)$

III $\quad y = (x - 1)^3 + 1$

IV $\quad y = (x + 1)(1 - x)(x + 3)$

(a) Which of these curves have negative y-intercepts?

(b) Which of the given curves has three distinct (different) roots?

Calculator Free

6. (c) Which of the given curves has two turning points?

(d) Which of the given curves has one turning point?

(e) Which of the given curves has no turning point?

---

7. [6 marks: 2, 4]

Find all possible equations of a cubic curve with:

(a) roots $x = 1, 2, -3$ and vertical intercept $(0, 12)$.

(b) exactly two roots at $x = -2$ and $x = 4$ and vertical intercept $(0, 16)$

Calculator Free

8. [12 marks: 2, 3, 2, 5]

Consider the cubic equation $y = f(x) = k(x + 2)(x^2 - 3x + c)$ where k and c are constants.

(a) Find the value of c if $f(4) = f(-2) = f(-1) = 0$.

(b) Find the value(s) of c if the cubic curve has three roots.

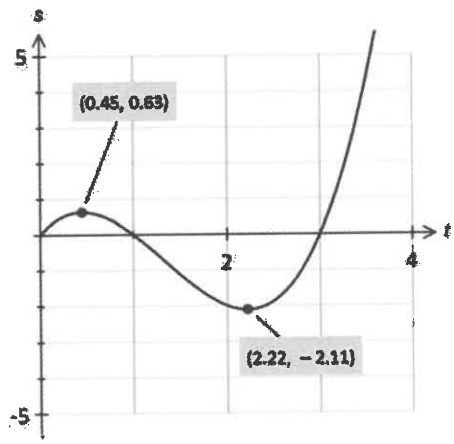
(c) Find the value(s) of c if the cubic has exactly two roots.

(d) Find the values of k and c if $f(-4) = f(-2) = 0$ and $f(0) = -4$.

Calculator Assumed

9. [8 marks: 2, 1, 2, 3]

The displacement, s metres, t seconds after a particle passes a fixed point O, is given by $s = t^3 - 4t^2 + 3t$, for $0 \le t \le 4$. The graph of s against t is given below. The graph has turning points at (0.45, 0.63) and (2.22, −2.11).



(a) Find when the particle returns to O.

(b) Find the displacement of the particle when $t = 2$.

(c) Find the farthest distance out from O reached by the particle in the interval $0 \le t \le 1$.

(d) Find the distance travelled by the particle in the first 2 seconds.

Calculator Assumed

10. [9 marks: 1, 2, 2, 4]

A particle P moves along a straight line. Its displacement t seconds after passing a fixed point O is given by $s = 0.001t(t-10)(t-40)$ metres, for $0 \le t \le 50$ seconds. Graph s against t on your graphic calculator. Use an appropriate routine to find:

(a) the displacement of P when $t = 50$ seconds.

(b) the farthest P is from O for $0 \le t \le 10$ seconds.

(c) the farthest P is from O for $10 \le t \le 40$ s

(d) the total distance travelled in the first 50 seconds.

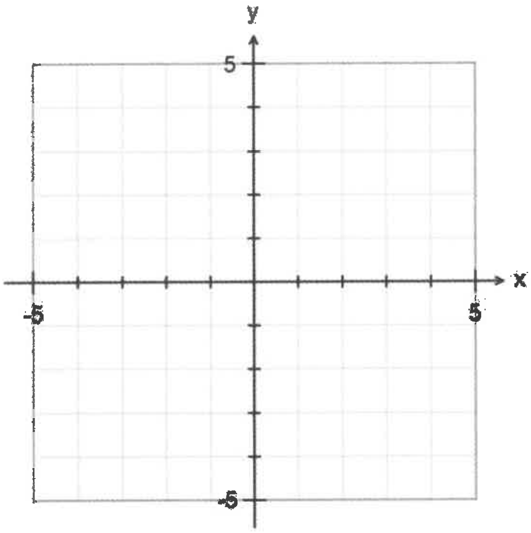
04 Rectangular Hyperbolas

Calculator Free

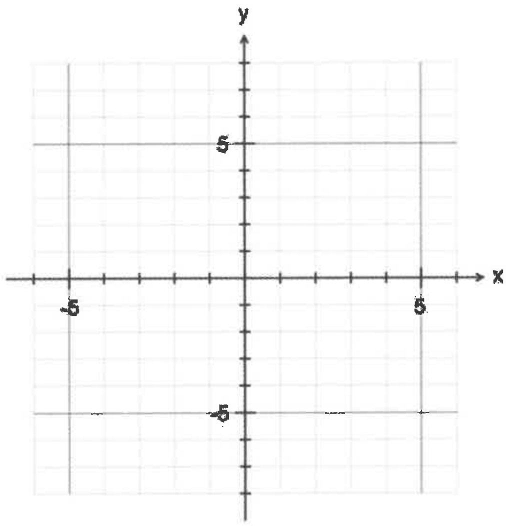
1. [4 marks: 2, 2]

Sketch in the axes provided, the graph of y against x. Show clearly all intercepts (if any) and asymptotes (if any).

(a) $y = \dfrac{2}{x}$ for $-4 \le x \le 4$.



(b) $y = -\dfrac{4}{x-1}$ for $-3 \le x \le 5$



2. [6 marks]

Complete the table below for the following curves.

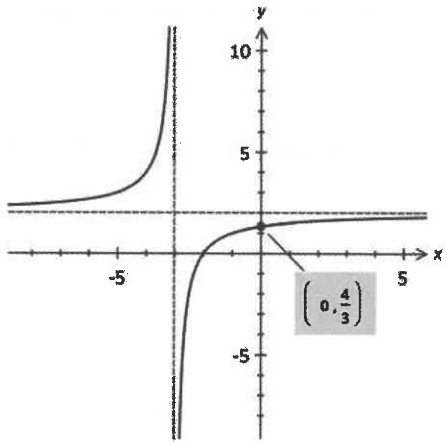
Curve	x-intercept	y-intercept	Asymptote parallel to the	
			x-axis	y-axis
$y = \dfrac{4}{x-3}$				
$y = \dfrac{-3}{2x+9}$				
$y = \dfrac{5}{2-x}$				
$y = \dfrac{15}{x+5} - 3$				
$(x+2)\,y = 10$				
$(x-1)(y-2) = 10$				

Calculator Free

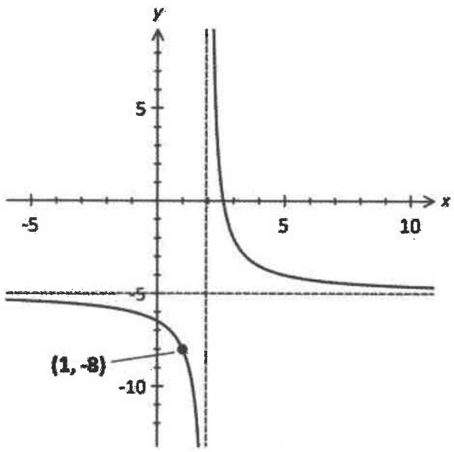
3. [9 marks: 3, 3, 3]

Find the equation of each of the following rectangular hyperbola.

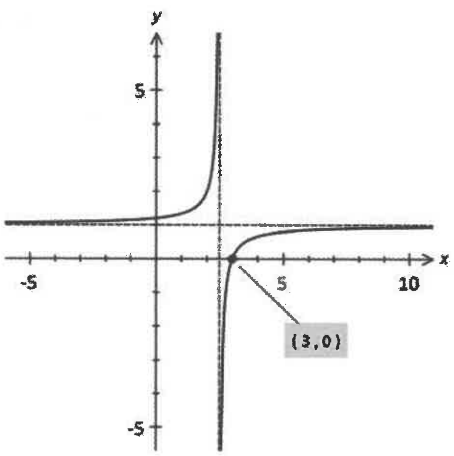
(a)



(b)



(c)



Calculator Free

4. [3 marks: 1, 1, 1]

Consider the following curves:

I $y = -\frac{1}{x}$ II $y = \frac{2}{x}$ III $y = \frac{1}{2x}$ IV $xy = -4$

(a) Which of the given curves passes through the point (–1, –2)?

(b) Which of the given curves has the property that when x is positive, y is negative?

(c) Which of the given curves has the property that as the value of x increases, the value of y decreases?

---

5. [8 marks: 2, 2, 2, 2]

A rectangular hyperbola has asymptotes with equation $x = -2$ and $y = 4$.

(a) Write two possible equations for this function.

(b) Write the equation of this function if it has a y-intercept at (0, 5).

(c) Write the equation of this function if it has a x-intercept at (–3, 0).

(d) Write the equation of this function if it passes through the point (3, 5).

Calculator Free

6. [8 marks: 2, 2, 2, 2]

Find y in terms of x if:

(a) y is inversely proportional to $2x - 5$ and $y = 8$ when $x = 4$.

(b) y is directly proportional to $\frac{1}{x}$ and $y = 5$ when $x = 20$.

(c) y is directly proportional to $\frac{1}{x+6}$ and $y = -2$ when $x = 4$.

(d) y is inversely proportional to x^3 and $y = 80$ when $x = 2$.

Calculator Free

7. [4 marks: 3, 1]

P is directly proportional to x and inversely proportional to y.
If P = 5 when $x = 1$ and $y = 4$, find:
(a) P in terms of x and y.

(b) y when P = 100 and $x = 20$.

---

8. [5 marks: 3, 2]

P is directly proportional to x^2 and inversely proportional to y^3.
If P = 20 when $x = 4$ and $y = 2$, find:
(a) P in terms of x and y.

(b) x when P = 2 and $y = 5$.

---

9. [3 marks]

T varies directly with the square root of I and varies inversely with the square root of M. Given that T = 2π, when I = 9 and M = 4. Find T in terms of I and M.

Calculator Assumed

10. [3 marks: 1, 1, 1]

A task can be completed by 8 workers in 6 days.

(a) How many workers would be required to complete the same task in half the time?

(b) How many days would double the number of workers take to complete the same task?

(c) What is the minimum number of workers required to complete the task in 12 days?

---

11. [3 marks: 2, 1]

At constant temperature, the volume of a gas (V m^3) varies inversely as its pressure (P pascals). If at a fixed temperature, 100 m^3 of a gas exerts a pressure of 200 pascals, find the:

(a) pressure exerted when the volume is 500 m^3.

(b) volume of the gas if the pressure exerted is 20 pascals.

---

12. [3 marks: 2, 1]

The amount of current (amps) flowing through an electrical circuit is inversely proportional to its resistance (ohms). If the current flow is 5 amps when the resistance is 8 ohms, find the:

(a) current flow when the resistance is 0.5 ohms.

(b) resistance when the current flow is 15 amps.

Calculator Assumed

13. [4 marks]

A food drop can feed 24 hikers for 6 whole days. Assuming the daily rations per hiker remains constant and given that there were at least 6 hikers, what are the possible numbers of hikers if the food is to last at least 10 whole days. Justify your answer.

---

14. [6 marks]

A project can be completed by 18 workers in 5 weeks. The same task is to be completed in exactly a whole number of weeks. How many workers would be required to achieve this? State all the possible combinations.

05 Exponential Functions I

Calculator Free

1. [7 marks: 1, 1, 2, 3]

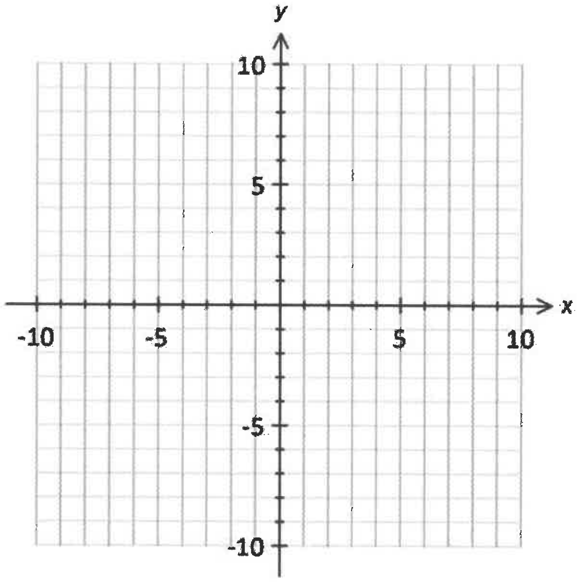
Consider the curve with equation $y = 2^x - 4$.

(a) State the equation of the horizontal asymptote of this curve.

(b) Find the coordinates of the vertical intercept of this curve.

(c) Find the coordinates of the horizontal intercept of this curve.

(d) On the axes provided below, sketch this curve.
Indicate clearly the intercepts and the asymptote(s).



Calculator Free

2. [7 marks: 1, 1, 2, 3]

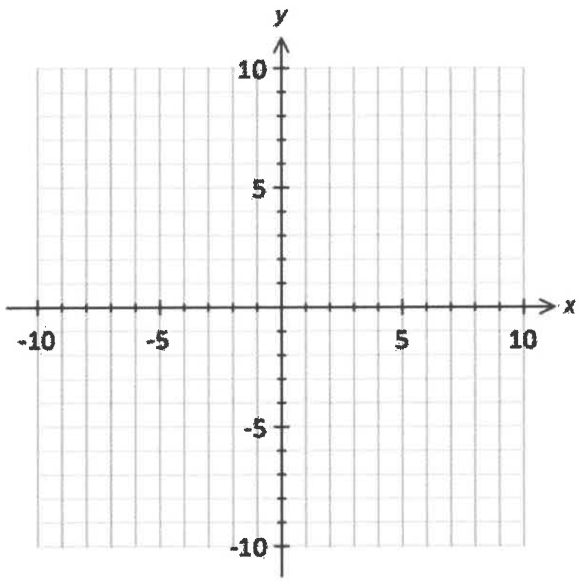
   Consider $y = 4 - 3^x$

   (a) State the equation of the horizontal asymptote of this curve.

   (b) Find the coordinates of the vertical intercept of this curve.

   (c) Find the point of intersection between this curve and the line $y = -5$.

   (d) On the axes provided below, sketch this curve.



Calculator Free

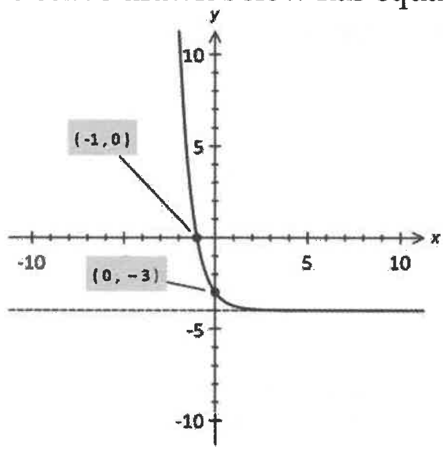
3. [4 marks: 2, 2]

(a) State two possible equations for an exponential curve with asymptote $y = -2$ and vertical intercept (0, –1).

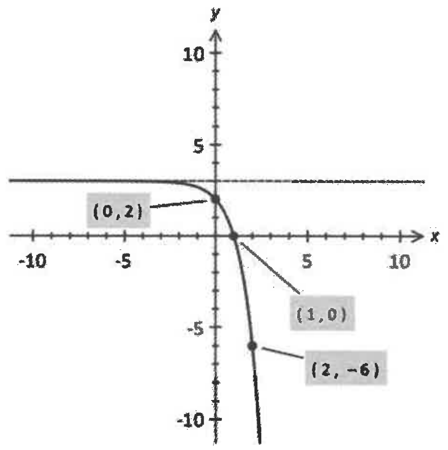
(b) State two possible equations for an exponential curve with asymptote $y = 2$ and vertical intercept (0, –3).

4. [7 marks: 2, 5]

(a) The curve drawn below has equation of the form $y = a^{-x} + b$. Find a and b.



(b) The curve drawn below has equation of the form $y = ka^x + b$. Find a, b and k.



06 Square Root Functions

Calculator Free

1. [9 marks: 2, 2, 2, 3]

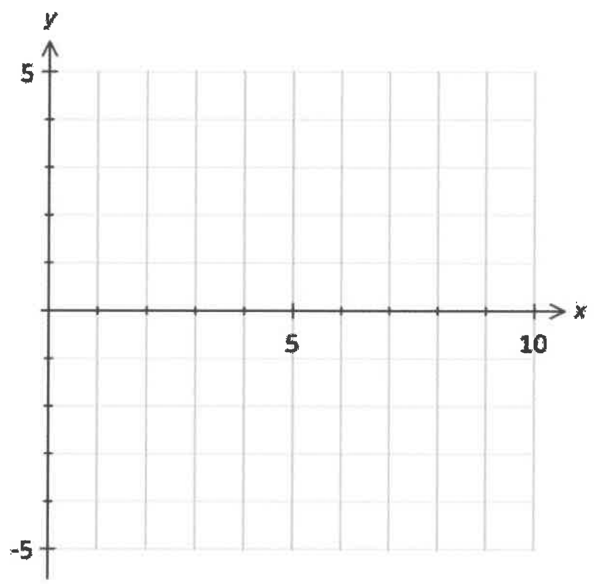
Consider the curve with equation $y = \sqrt{x-4}$.

(a) Explain why it is not possible for this curve to exist for values of $x < 4$.

(b) Find the coordinates of the horizontal intercept of this curve.

(c) Determine the point of intersection between this curve and the line $y = 2$.

(d) On the axes provided below, sketch this curve.



Calculator Free

2. [9 marks: 2, 2, 2, 3]

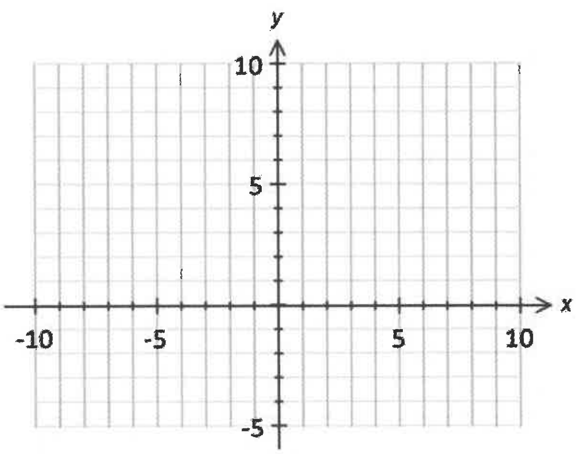
Consider the curve with equation $y = 2 + \sqrt{x+9}$.

(a) Explain why $y \geq 2$.

(b) Find the coordinates of the vertical intercept of this curve.

(c) Determine the point of intersection between this curve and the line $y = 6$.

(d) On the axes provided below, sketch this curve.



Calculator Free

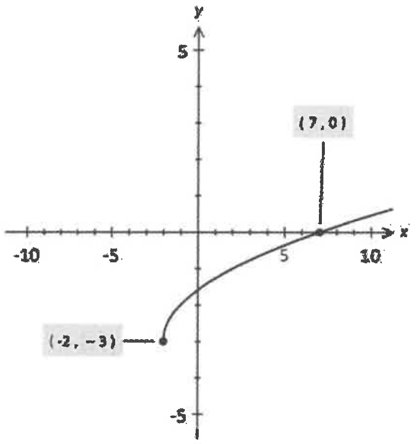
3. [4 marks: 2, 2]

(a) State two possible equations for the curve with equation $y = a + k\sqrt{x+b}$ if the curve has $x \le 2$ and $y \le -3$.

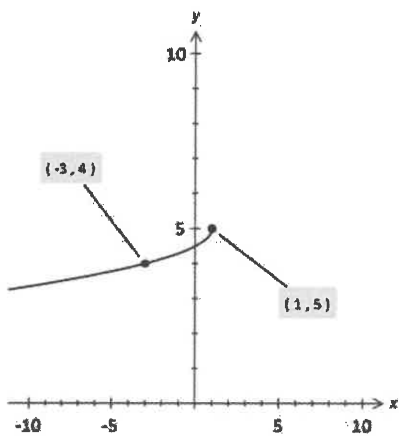
(b) State two possible equations for the curve with equation $y = a + k\sqrt{x+b}$ if the curve has $x \ge -3$ and $y \ge 5$.

4. [6 marks: 3, 3]

(a) Find the equation of the curve drawn below with equation $y = a + k\sqrt{x+b}$.



(b) Find the equation of the curve drawn below with equation $y = a + k\sqrt{x+b}$.

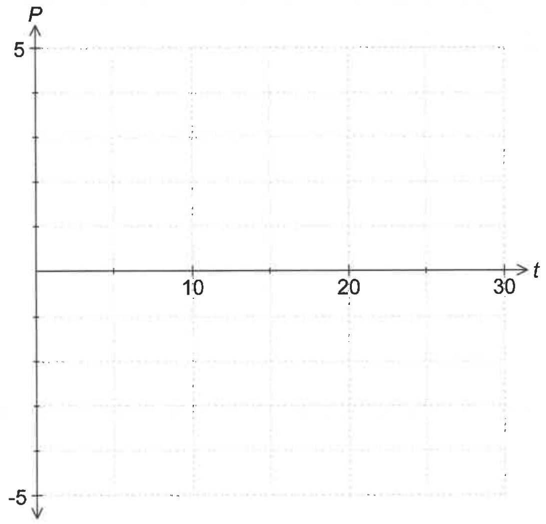


Calculator Assumed

5. [8 marks: 2, 2, 2, 2]

The daily Profit (in hundreds of dollars) for a small Lunch Bar is modelled by $P = -1 + \sqrt{t-5}$ for $5 \le t \le 30$, where t is time in days after 1st July.

(a) Sketch P against t in the axes provided below. Show clearly all essential features of the graph.



(b) On what date did the Lunch Bar open for Business and what was the profit for that day?

(c) How many days did the Lunch Bar take to make its first profit?

(d) What was the profit, three weeks after the Lunch Bar first opened.

07 Circles & Parabolas

Calculator Free

1. [9 marks: 2, 2, 2, 3]

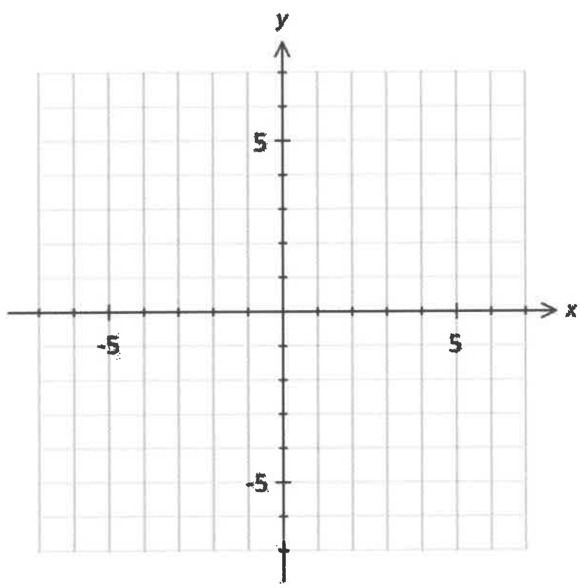
Consider the circle with equation $(x-1)^2 + (y-3)^2 = 10$.

(a) Find the coordinates of the x-intercepts.

(b) Find the coordinates of the y-intercepts.

(c) Determine the coordinates of the centre of this circle and its radius.

(d) On the axes provided below, sketch this circle.



Calculator Free

2. [10 marks: 2, 3, 2, 3]

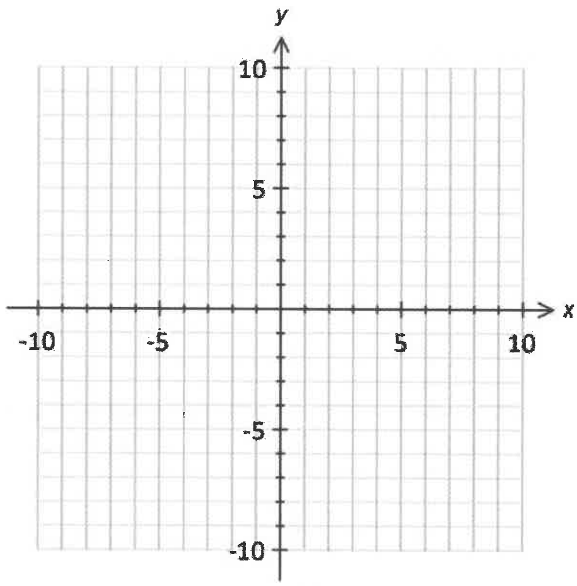
Consider the circle with equation $(x + 2)^2 + (y + 3)^2 = 25$.

(a) Find the coordinates of the x-intercepts.

(b) Find the coordinates of the y-intercepts.

(c) Determine the coordinates of the centre of this circle and its radius.

(d) On the axes provided below, sketch this circle.



Calculator Free

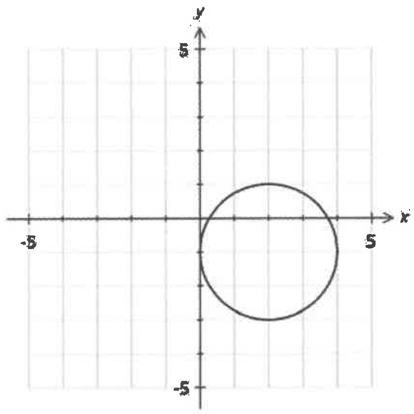
3. [4 marks]

State two possible equations for a circle with radius 5 and passing through the point with coordinates (4, 4).

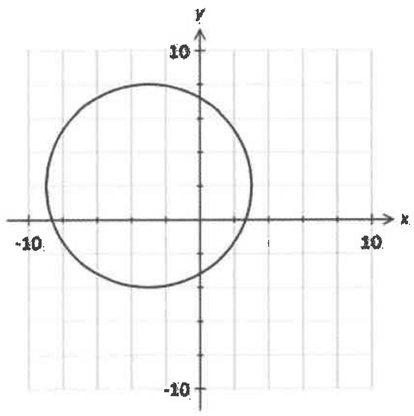
---

4. [6 marks: 3, 3]

(a) Find the equation of the circle drawn below.



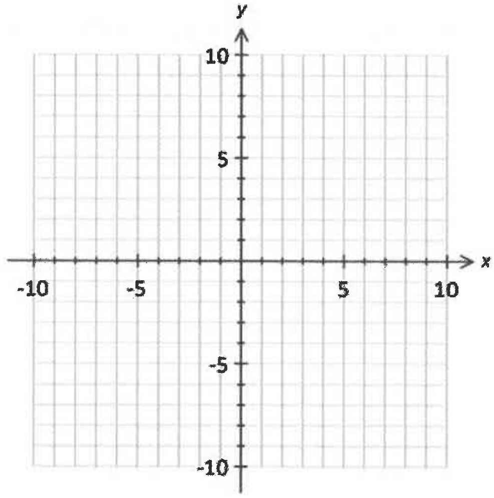
(b) Find the equation of the circle drawn below.



Calculator Free

5. [3 marks]

On the axes provided, sketch the parabola with equation $y^2 = 16x$.



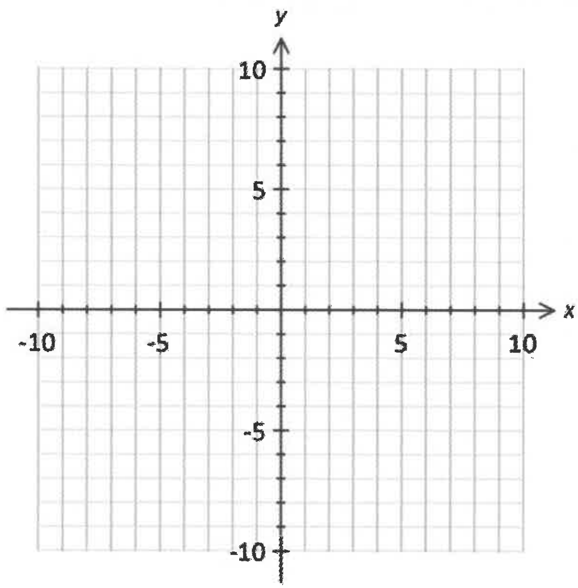
---

6. [6 marks: 3, 3]

Consider the parabola with equation $y^2 = 3(x + 3)$.

(a) State the coordinates of the x and y intercepts.

(b) On the axes provided sketch this parabola.



Calculator Free

7. [4 marks: 2, 2]

State a possible equation for a parabola passing through the point (1, 4):

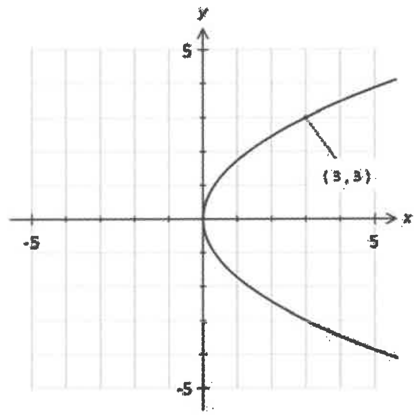
(a) symmetrical about the y-axis.

(b) symmetrical about the x-axis.

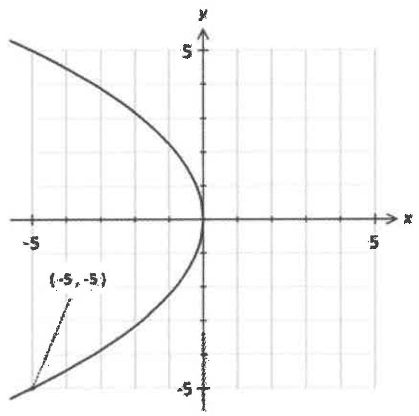
8. [4 marks: 2, 2]

Find the equation of the parabola drawn below.

(a)



(b)



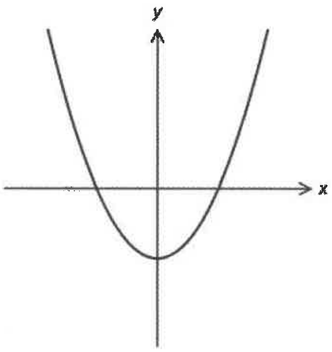
08 Functions & Relations I

Calculator Free

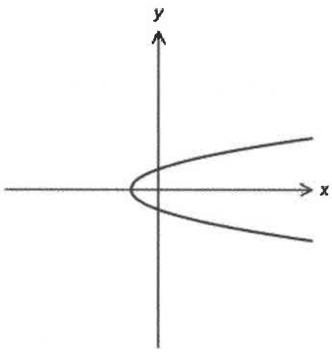
1. [6 marks: 1 each]

Determine with reasons if each of the following graphs represent functions or relations.

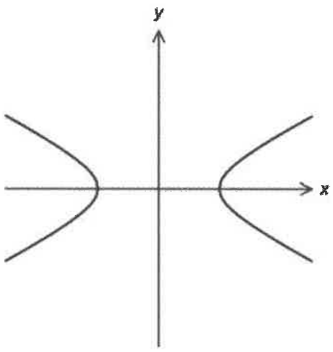
(a)



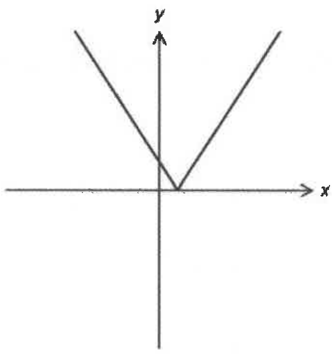
(b)



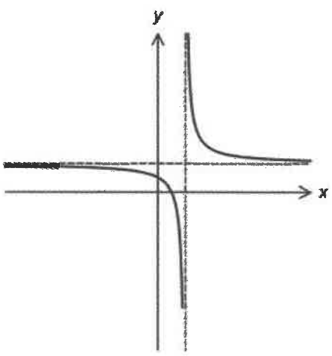
(c)



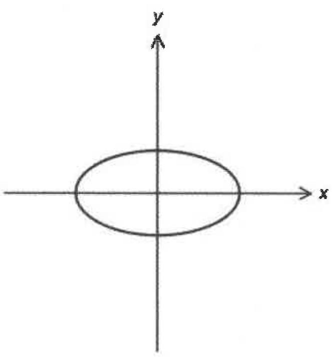
(d)



(e)



(f)

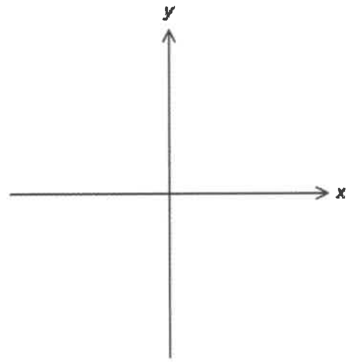


Calculator Free

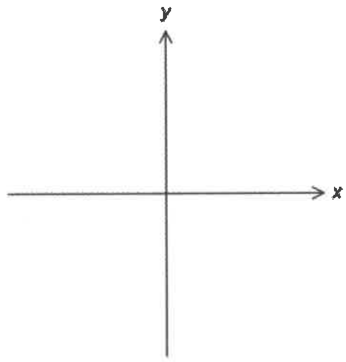
2. [9 marks: 2, 2, 2, 3]

In the axes provided, make a sketch of:

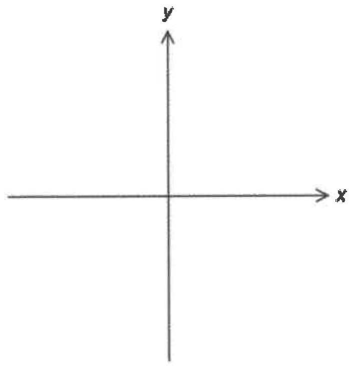
(a) the graph of a function which has a horizontal asymptote.



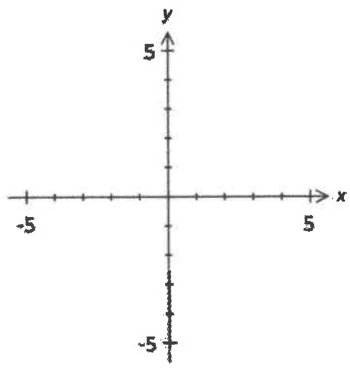
(b) the graph of a relation that is not symmetrical about the x-axis.



(c) the graph of a function that exists only for certain values of x.



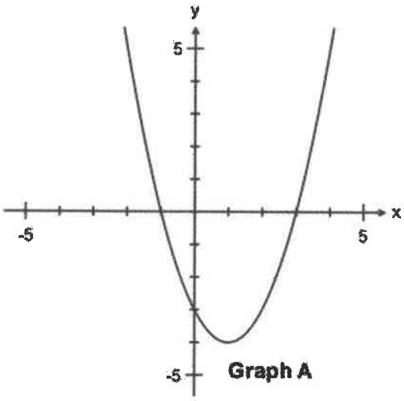
(d) the graph of a relation which is not symmetrical about the y-axis but symmetrical about the x-axis.

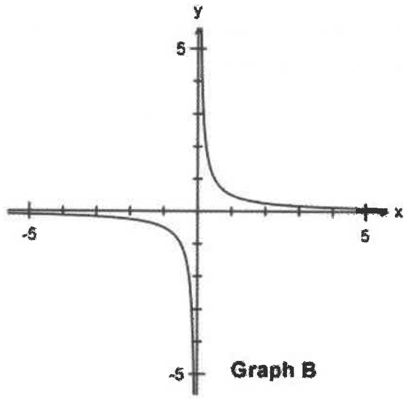


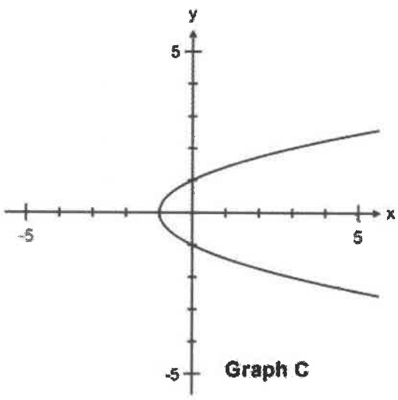
Calculator Free

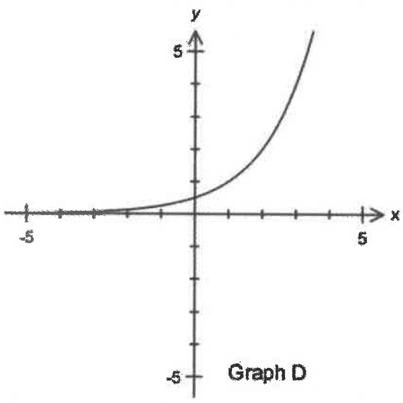
3. [4 marks: 1 each]

Match each of the following graphs with an equation from the given list.









Equation I: $y = \dfrac{1}{2x}$ Equation II: $y = x^2 - 2x - 3$

Equation III: $y = 2^{x-1}$ Equation IV: $y^2 = x - 1$

Equation V: $y = (x + 1)^2 - 4$ Equation VI: $x = y^2 - 1$

Equation VII: $y = 2^x$ Equation VII: $y = \dfrac{1}{x}$

Graph	Equation
A	
B	
C	
D	

Calculator Free

4. [5 marks]

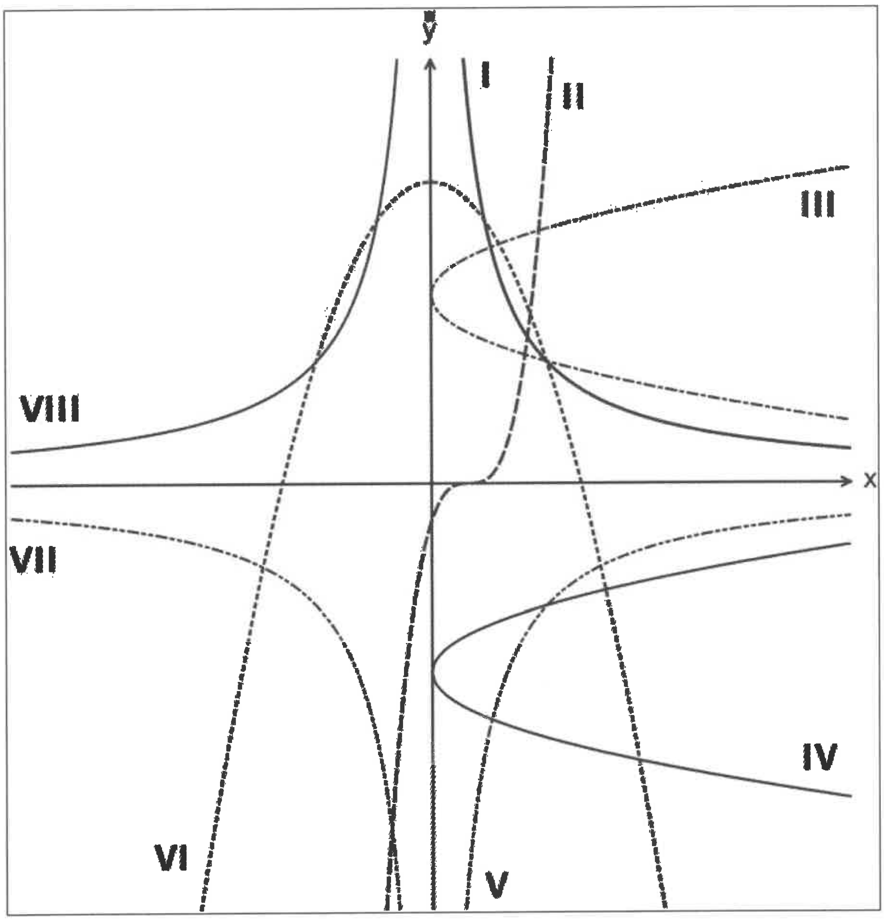
Match each of the following equations with one (or more) of the given curves.

Equation A: $y = \frac{1}{2}(16 - x^2)$

Equation B: $y = x^3 - 3x^2 + 3x - 1$

Equation C: $y = \frac{10}{x}$

Equation D: $x = (y + 5)^2$

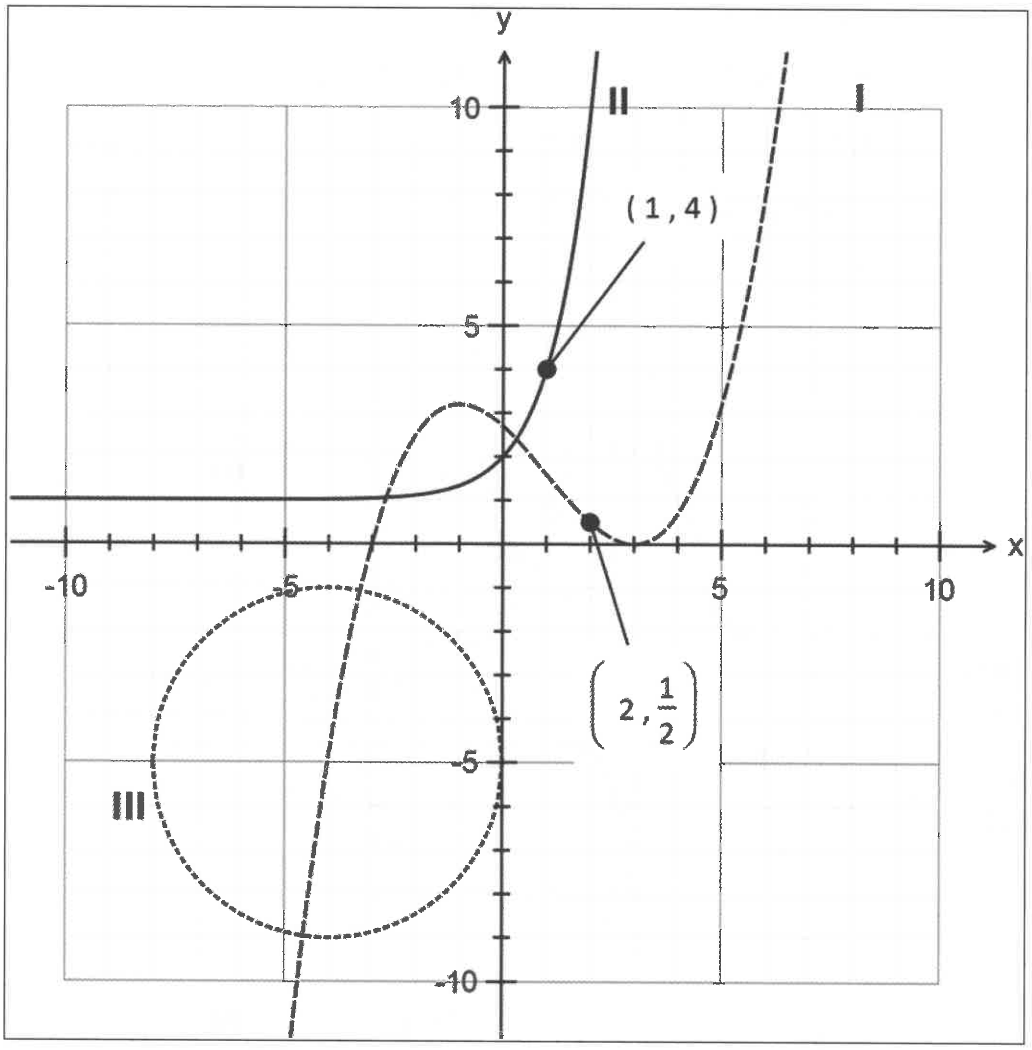


Equation	Graph
A	
B	
C	
D	

Calculator Free

5. [8 marks]

Find the equation of the curves labelled I, II and III:

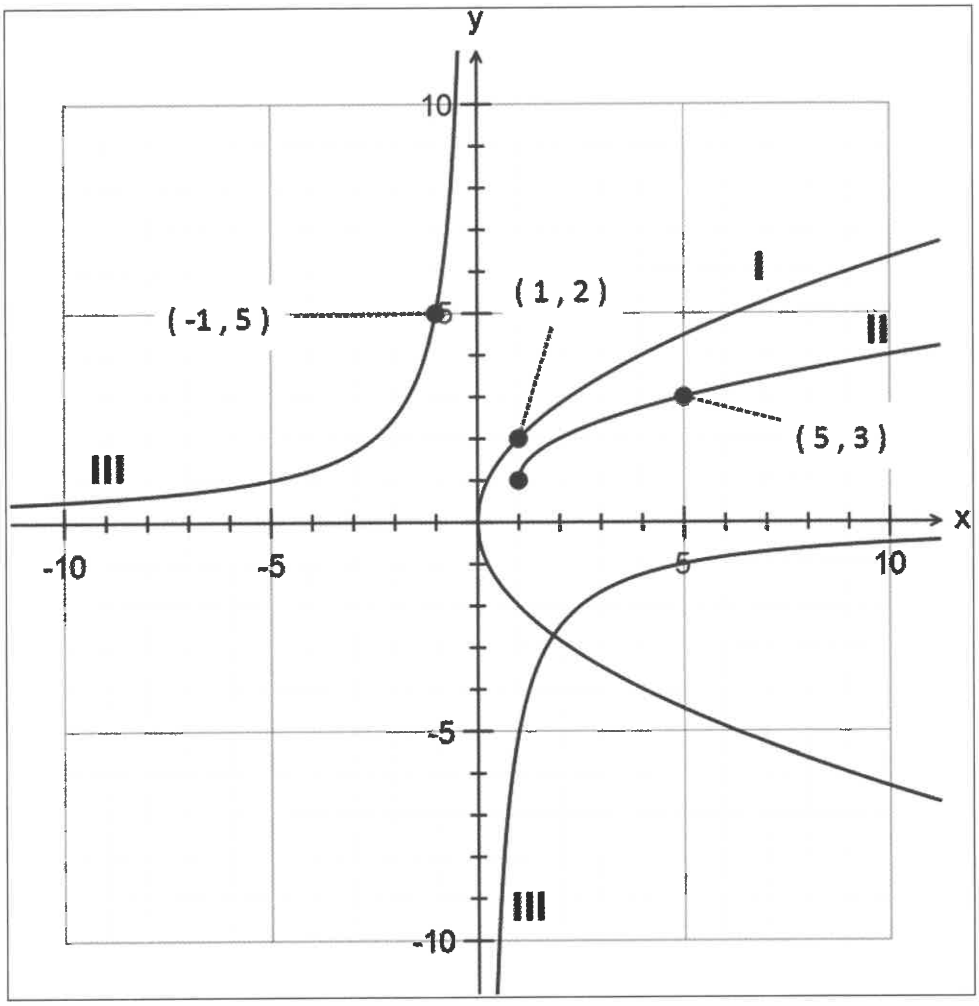


Curve	Equation
I	
II	
III	

Calculator Free

6. [6 marks]

Find the equation of the curves labelled I, II and III:



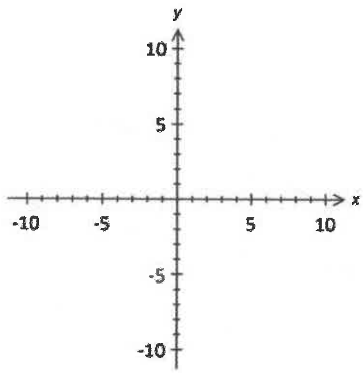
Curve	Equation
I	
II	
III	

Calculator Free

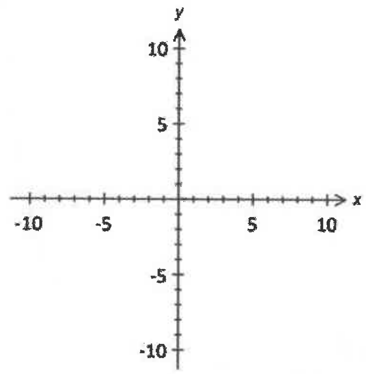
7. [15 marks: 2, 3, 2, 3, 2, 3]

Sketch each of the following. Indicate clearly all intercepts, asymptotes and symmetries where appropriate.

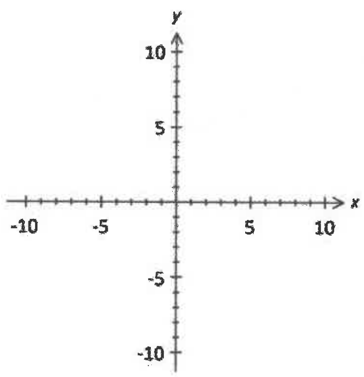
(a) $4x + 5y = 20$



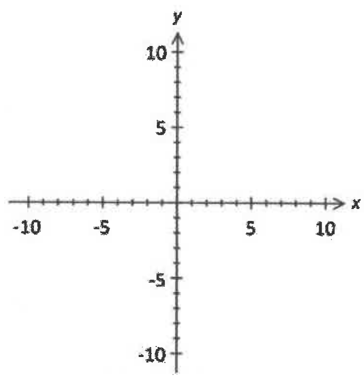
(b) $y = -0.5(x + 4)(x - 3)$



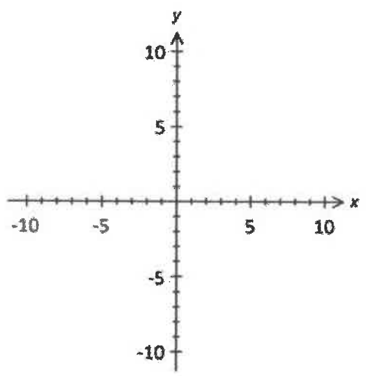
(c) $y = \dfrac{4}{x-2}$



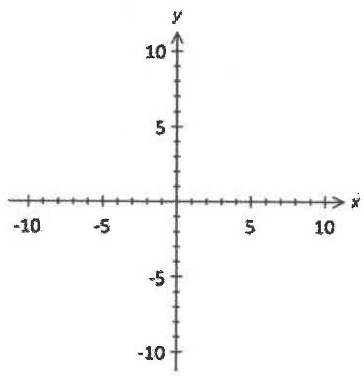
(d) $y = 0.2(x^2 - 4)(x - 5)$



(e) $y^2 = 4(x - 1)$



(f) $(x - 2)^2 + (y + 3)^2 = 25$

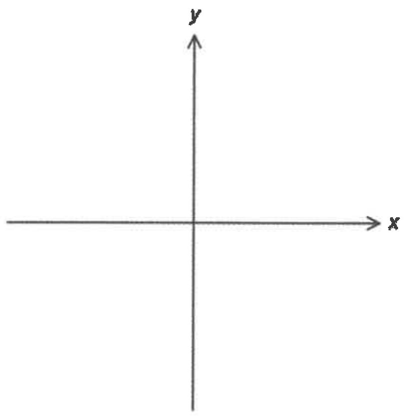


Calculator Free

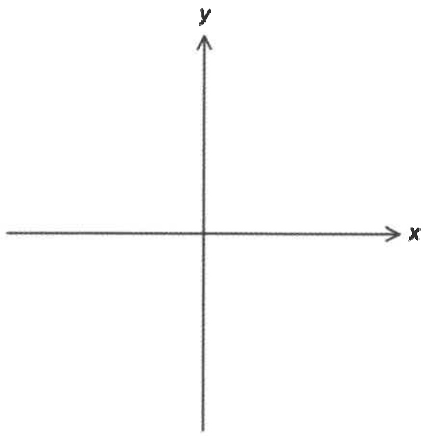
8. [6 marks: 2, 2, 2]

In the axes provided sketch:

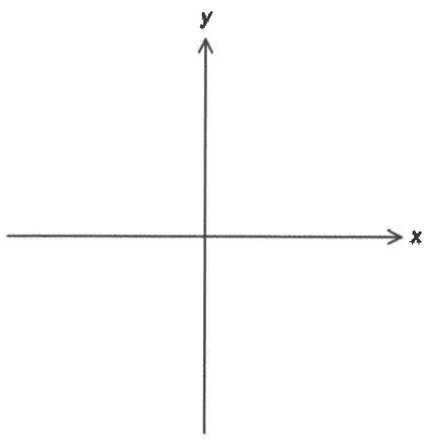
(a) a line with negative gradient and a positive y-intercept.



(b) a parabola with a positive y-intercept with no roots.



(c) a reciprocal function where the x and y values have opposite signs.

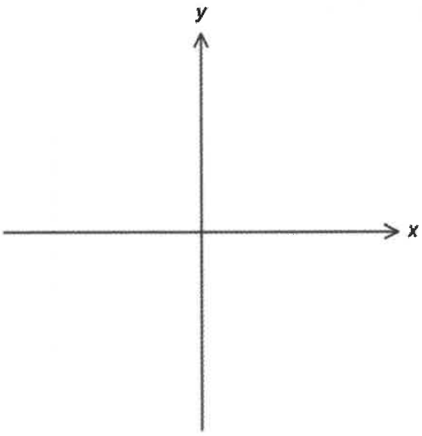


Calculator Free

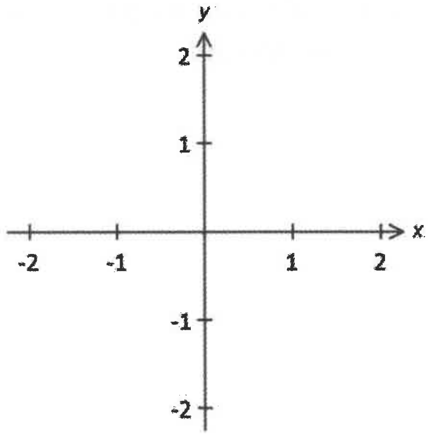
9. [12 marks: 4, 4, 4]

(a) Make a sketch of $ax + by = c$ where a, b and c are constants if:

(i) $a < 0$ and $b = 0$ and $c > 0$

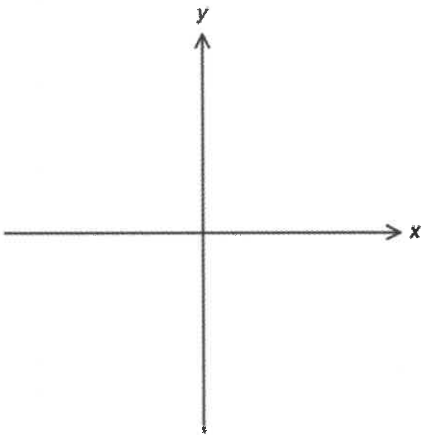


(ii) $a = b = c$

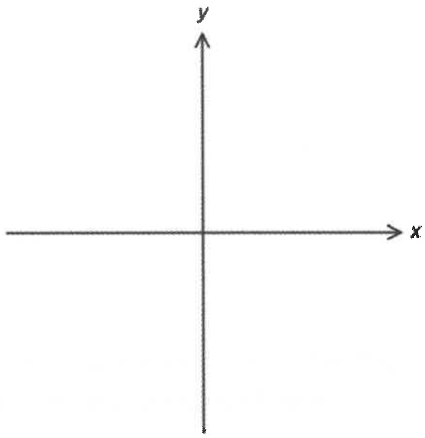


(b) Make a sketch of $y = ax^2 + bx + c$ where a, b and c are constants if:

(i) $a < 0$ and $b = 0$ and $c = 0$

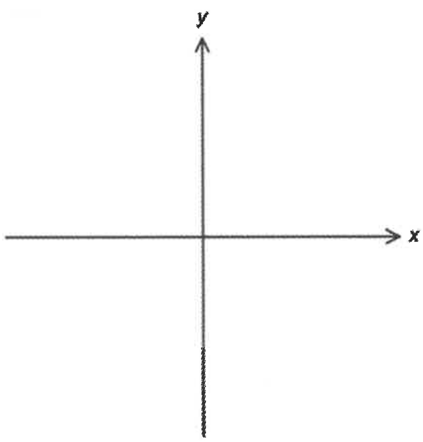


(ii) $a > 0$ and $b^2 = 4ac$

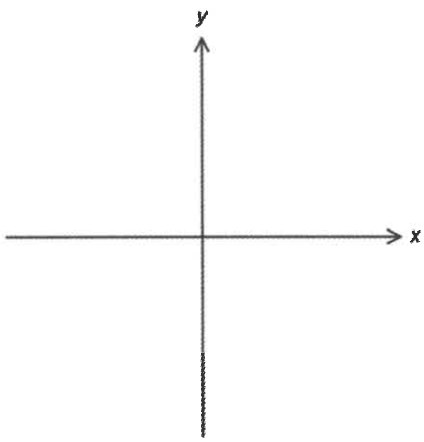


(c) Make a sketch of $y = k(x + a)(x + b)(x + c)$ where k, a, b and c are constants if:

(i) $k < 0$ and $a = b = c = 0$



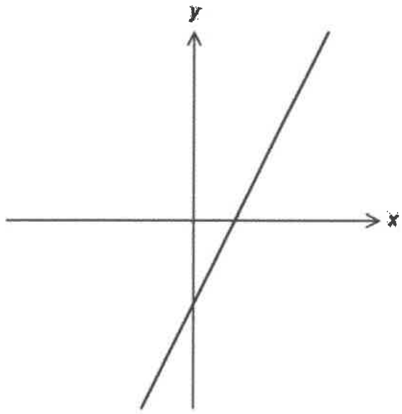
(ii) $k > 0$ and $a = -b$ and $c = 0$



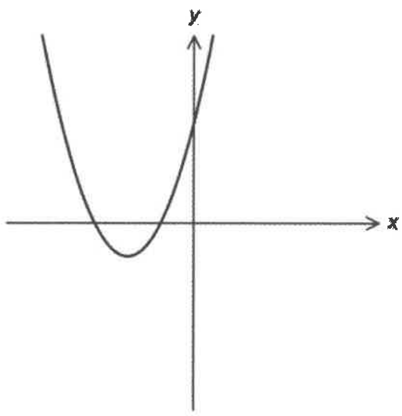
Calculator Free

10. [7 marks: 2, 3, 2]

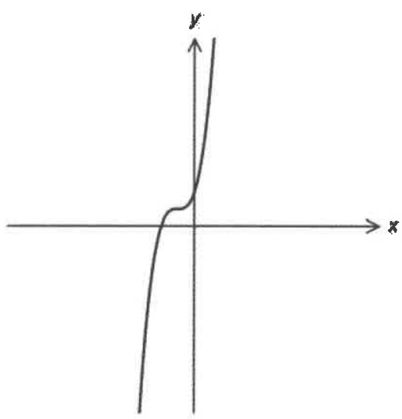
(a) The graph of $ax + by = c$ where a, b and c are constants is given in the accompanying diagram. If $a < 0$ and $b > 0$ determine with reasons if c is positive or negative.



(b) The graph of $y = ax^2 + bx + c$ where a, b and c are constants is shown in the accompanying diagram. Explain clearly why $a > 0$, $b > 0$ and $c > 0$.



(c) The graph of $y = k(x + m)(ax^2 + bx + c)$ where k, m, a, b and c are constants is shown in the accompanying diagram. Explain clearly why $b^2 - 4ac < 0$.



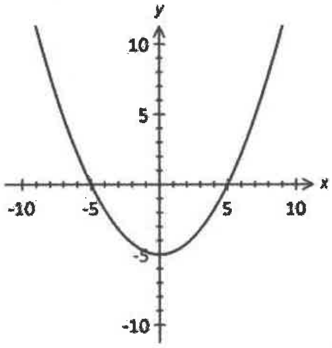
09 Functions & Relations II: Domain & Range

Calculator Free

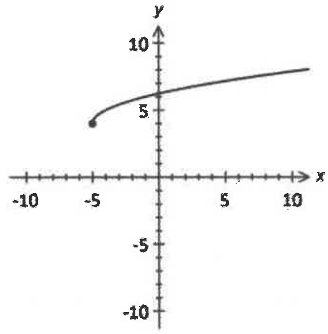
1. [12 marks: 2, 2, 2, 2, 2, 2]

   The graphs of several relations/functions are shown in the accompanying diagrams. In each case, state the domain and range for each relation/function.

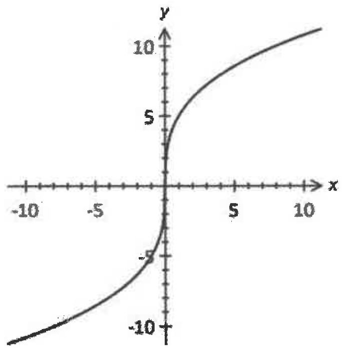
(a)



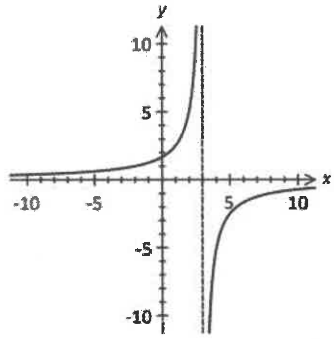
(b)



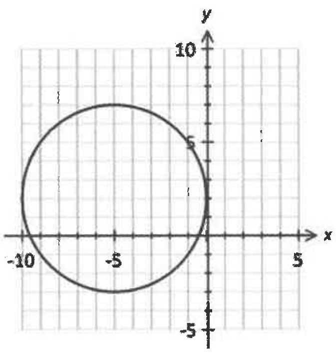
(c)



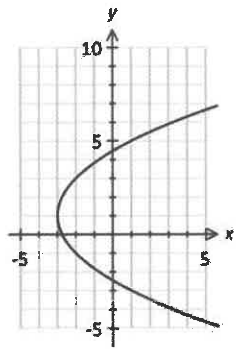
(d)



(e)



(f)



Calculator Free

2. [20 marks: 1 each]

State the natural domain and range for each of the relations/functions below.

Function	Natural Domain	Natural Range
$y = (x+1)^2 - 5$		
$y = 4 - 2(3x-1)^2$		
$y = \sqrt{x-5}$		
$y = \sqrt{x+3} - 10$		
$y = 5^x + 3$		
$y = -4 - 2^x$		
$y = \frac{1}{x-1} + 3$		
$y = 5 - \frac{3}{2x-4}$		
$(x+1)^2 + (y+1)^2 = 4$		
$y^2 = 4(x-1)$		

Calculator Free

3. [12 marks: 3, 2, 2, 2, 3]

Consider the function with equation $y = \sqrt{16 - x^2}$.

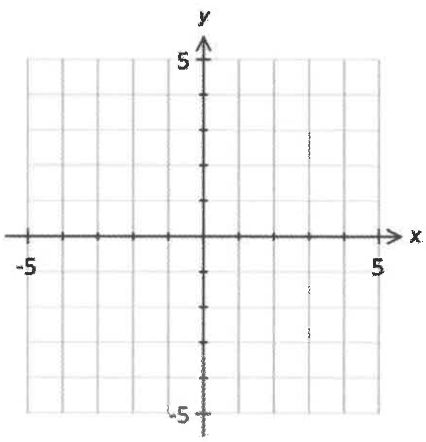
(a) The coordinates of the x-intercepts of this curve are $(a, 0)$ and $(b, 0)$ where $a \le b$. Find a and b.

(b) Explain why this curve only exists for values of x in the interval $a \le x \le b$.

(c) What is the minimum and maximum value of y?

(d) Determine the domain and range of this function.

(e) On the axes provided, sketch this curve.



Calculator Free

4. [10 marks: 2, 1, 2, 2, 3]

Consider the function with equation $y = 5 + \sqrt{9 - x^2}$.

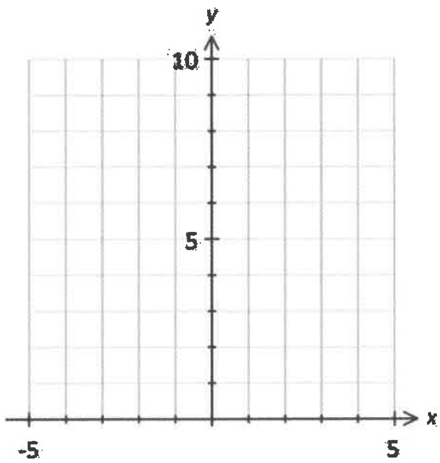
(a) Explain why this curve exists only for $-3 \leq x \leq 3$.

(b) Explain why the y-value must always be at least 5.

(c) What is the largest possible value of y?

(d) Determine the domain and range of this function.

(e) On the axes provided, sketch this curve.



Calculator Assumed

5. [9 marks: 3, 3, 3]

Consider the function $f(x) = x + 1$.

(a) Express in terms of x, $y = f(x^3)$.
Hence, find the domain and range for $y = f(x^3)$.

(b) Express in terms of x, $y = f((x-1)^2)$.
Hence, find the domain and range for $y = f((x-1)^2)$.

(c) Express in terms of x, $y = f(\sqrt{x+2})$.
Hence, find the domain and range for $y = f(\sqrt{x+2})$.

---

6. [4 marks]

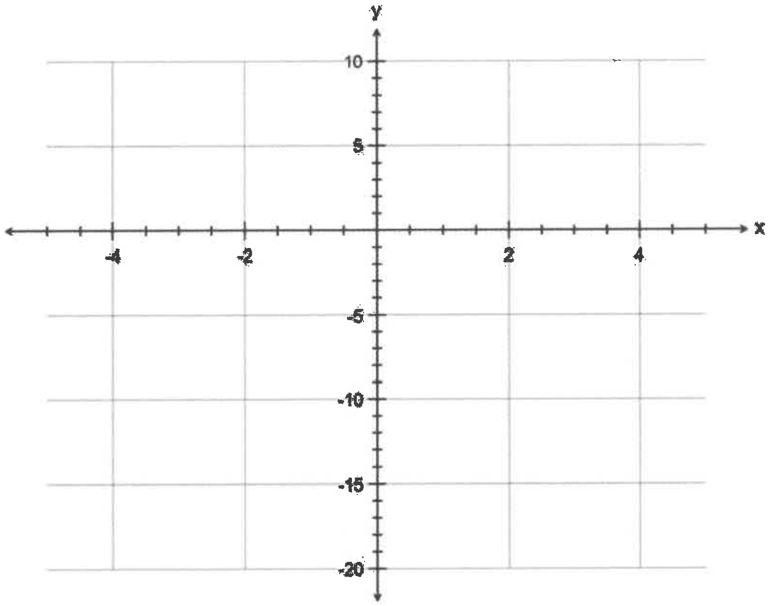
Consider the function $f(x) = \sqrt{4-x}$.
Express in terms of x, $y = f(2^x)$. Hence, find the domain and range for $y = f(2^x)$.

Calculator Assumed

7. [9 marks: 3, 2, 2, 2]

Consider the function $f(x) = x^3 - 3x^2 - x + 3$ for $-2 \le x \le 2$.

(a) In the axes provided below, sketch the graph of $y = f(x)$ within the specified domain.



(b) State the range for $f(x)$ for the domain specified. Give your answer correct to one decimal place.

(c) State the coordinates of the horizontal intercept(s) of $y = f(x)$ for the domain specified.

(d) State the coordinates of the turning point(s) of $y = f(x)$ for the domain specified. State the nature of this point. Give your answer correct to one decimal place.

10 Transformations on Curves

Calculator Free

1. [10 marks: 2, 2, 2, 2, 2]

Describe a sequence of transformations required to convert $y = f(x)$ into $y = g(x)$.

(a) $f(x) = x^2$ and $g(x) = (x-2)^2 + 4$

(b) $f(x) = x^3$ and $g(x) = -(2x)^3$

(c) $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{1-x}$

(d) $f(x) = 3^x$ and $g(x) = -3^{x+1}$

(e) $f(x) = (2x+1)^2$ and $g(x) = x^2$.

---

2. [4 marks: 2, 2]

Describe a sequence of transformations required to transform:

(a) $x^2 + y^2 = 100$ into $(x+5)^2 + (y-6)^2 = 100$

(b) $(x-2)^2 + (y-1)^2 = 64$ into $(x+7)^2 + (y+3)^2 = 64$

Calculator Free

3. [4 marks: 2, 2]

The curve $y = 2^{x+1}$ is transformed into $y = g(x)$.

(a) State the sequence of transformations involved if $g(x) = 2^{0.5x-1}$.

(b) State the sequence of transformations involved if $g(x) = 3(2^x)$.

---

4. [4 marks: 2, 2]

The curve $y = 1 + \frac{1}{x-2}$ is transformed into $y = g(x)$.

(a) State the sequence of transformations involved if $g(x) = \frac{2}{x-2}$.

(b) State the sequence of transformations involved if $g(x) = -1 + \frac{1}{x+2}$.

Calculator Free

5. [10 marks: 2, 2, 2, 2, 2]

Identify the sequence of transformations required to map:

(a) $y = f(x)$ to $y = 2f(2x)$

(b) $y = f(x)$ to $y = f(2x + 1)$

(c) $y = f(x)$ to $y = f(2(x + 1))$

(d) $y = f(x)$ to $y = f(1 - x)$

(e) $y = f(x)$ to $y = 1 - f(x)$

Calculator Free

6. [6 marks: 2; 2, 2]

A parabola has equation $y = x^2 + 2x - 3$. Find the equation of the resulting curve:

(a) if the parabola is dilated by a factor of 2 along the x-axis.

(b) if the parabola is reflected about the x-axis and then translated 2 units along the negative y-axis.

(c) if the parabola is translated 1 unit along the positive x-axis and then reflected about the y-axis.

---

7. [6 marks: 2, 2, 2]

The curve $y = 5^x$ is mapped to $y = g(x)$ by the following sequence of transformations. Find $g(x)$.

(a) a translation in the direction of the positive x-axis by 3 units followed by a translation in the direction of the positive y-axis by 2 units

(b) a dilation in the direction of the positive x-axis by a factor of 2 followed by a translation in the direction of the positive x-axis by -2 units

(c) a reflection about the y-axis followed by a dilation in the direction of the positive x-axis by a factor of ½.

Calculator Free

8. [10 marks: 2, 2, 2, 2, 2]

A curve with equation $y = \sqrt{x}$ is transformed into $y = k\sqrt{(ax+b)} + c$ by the following sequences of transformations. State the values of k, a, b and c.

(a) A translation 5 units in the direction of the positive x-axis followed by a dilation parallel to the positive x-axis of factor 2.

(b) A dilation parallel to the positive x-axis of factor 2 followed by a translation 5 units in the direction of the positive x-axis.

(c) A translation 5 units in the direction of the negative y-axis followed by a reflection about the x-axis.

(d) A reflection about the x-axis followed by a translation 5 units in the direction of the negative y-axis

(e) A reflection about the y-axis followed by a dilation of factor 3 parallel to the positive y-axis.

Calculator Free

9. [4 marks: 2, 2]

The circle with equation $(x + 6)^2 + (y - 7)^2 = 81$ is transformed into the circle with equation $(x - a)^2 + (y - b)^2 = r^2$ by the following sequences of transformations. State the values of a, b and r.

(a) A translation 3 units in the direction of the positive x-axis followed by a translation 5 units in the direction of the negative y-axis.

(b) A dilation of factor 2 parallel to the x-axis followed by a dilation of factor 2 parallel to the y-axis.

---

10. [4 marks: 2, 2]

The parabola with equation $y^2 = x$ is transformed into the parabola with equation $y^2 = k(x - a)$ by the following sequences of transformations. State the values of a and k.

(a) A reflection about the y-axis followed by a reflection about the x-axis.

(b) A translation 4 units in the direction of the positive x-axis followed by a reflection about the y-axis.

Calculator Free

11. [14 marks: 3, 3, 4, 4]

The curve $y = f(x)$ has a minimum turning point at $(-2, -1)$ and a maximum turning point at $(4, 6)$. Find the minimum and maximum turning points of the following curves. In each case, explain clearly how you obtained your answer.

(a) $y = f(2x)$

(b) $y = 2f(x)$

(c) $y = 1 - f(x)$

(d) $y = f(1 - x)$.

Calculator Assumed

12. [8 marks: 2, 2, 2, 2]

The curve $y = f(x)$ has a maximum point at (1, 5), a minimum point at (–5, 2) and intercepts at (0, 4) and (5, 0). The curve has no other turning points and intercepts.

(a) State the coordinates of the horizontal intercept(s) of the curve $y = f(-x - 1)$.

(b) State the coordinates of a horizontal intercept of the curve $y = f(x + 1) - 2$.

(c) State the coordinates of the vertical intercept(s) of the curve $y = 2f(x + 1)$.

(d) State the coordinates of the maximum and minimum point of $y = -f(-x)$.

---

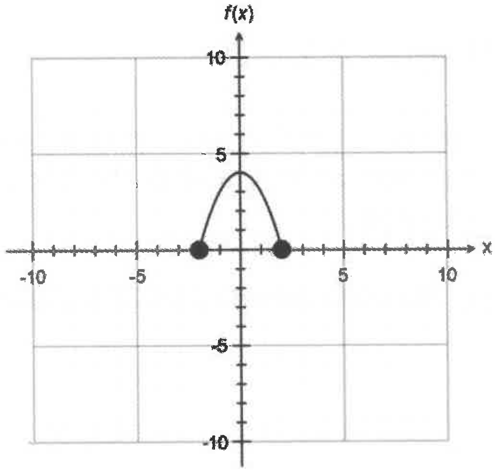
13. [3 marks]

Given that $f(x) = x^2$, solve $f(x) = f(2x + 1)$. Describe clearly how you obtained your answer.

Calculator Assumed

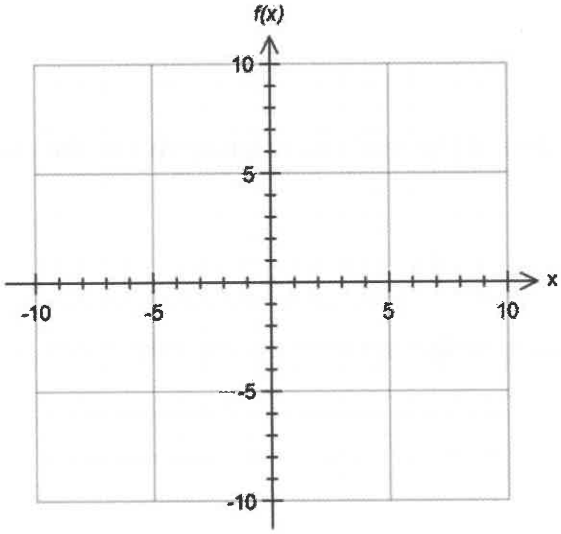
14. [6 marks: 3, 3]

The sketch of $y = f(x)$ is given in the accompanying diagram.

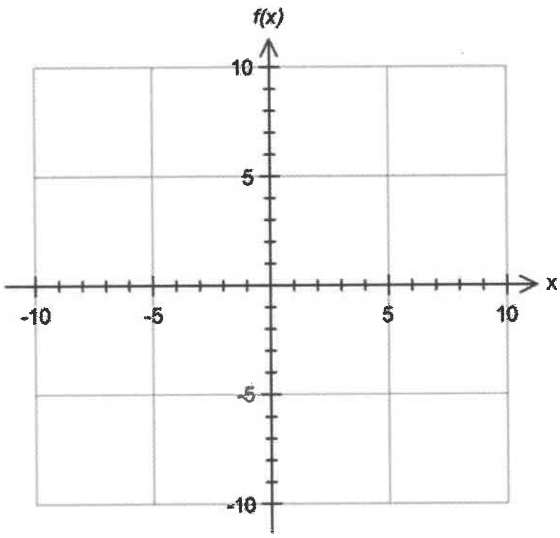


Sketch:

(a) $y = \frac{3}{2}f(x)$



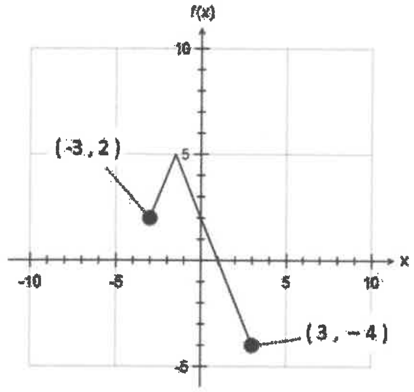
(b) $y = f(\frac{x}{2}+1)$



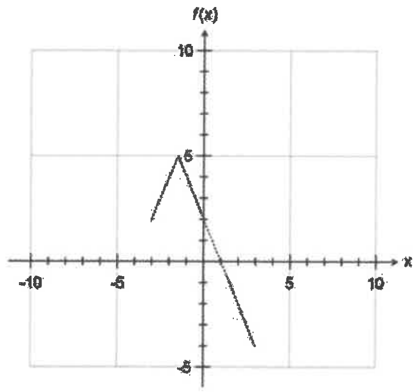
Calculator Assumed

15. [6 marks: 3, 3]

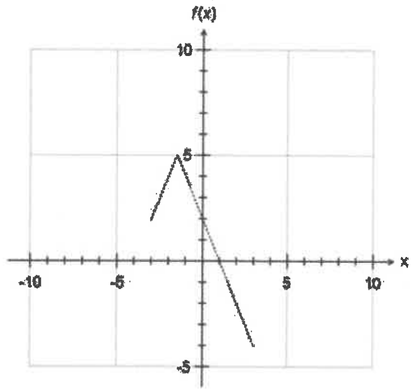
Given the graph of $y = f(x)$, sketch in the axes provided $y = g(x)$.



(a) $g(x) = \frac{1}{2}\left(f(x)+1\right)$

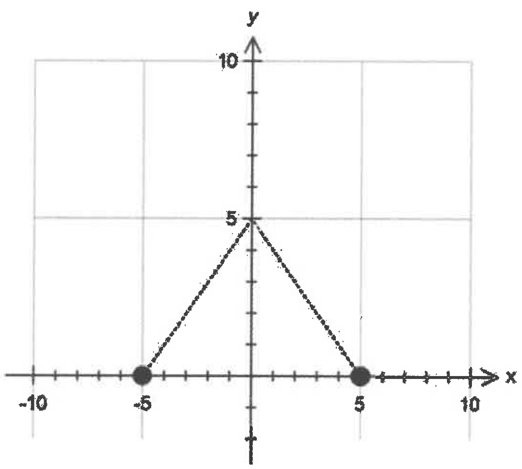


(b) $g(x) = f\left(\frac{1}{2}(x+1)\right)$



16. [3 marks]

Given the graph of $y = 5 - f(2x)$, sketch in the axes provided $y = f(x)$.



11 Equations

Calculator Free

1. [17 marks: 2, 2, 3, 3, 3, 4]

Solve for x:

(a) $2x - 5 = -3x + 4$

(b) $(2x - 5)(4 - 3x) = 0$

(c) $4x^2 - 49 = 0$

(d) $x^2 + 1 = 4x - 3$

(e) $(2x - 1)^2 - 25 = 0$

(f) $x^2 + 4x - 3 = 0$

Calculator Free

2. [20 marks: 2, 2, 1, 3, 3, 5, 4]

Solve for real values of x:

(a) $(x-5)(x+3)(1-4x)=0$

(b) $(x+3)(x^2-36)=0$

(c) $(x^2+1)(2x-5)=0$

(d) $(x^2-5x+6)(3-2x)=0$

(e) $x^3=x^2+2x$

(f) $x^3+4x^2-7x-10=0$

(g) $2x^3+5x^2-4x-3=0$

Calculator Free

3. [15 marks: 3, 3, 3, 3, 3]

Solve for x:

(a) $\frac{3}{x} = x + 2$

(b) $\frac{2}{x-1} = \frac{1}{x+4}$

(c) $\frac{-1}{x+1} = x + 3$

(d) $\frac{1}{x} = x + 1$

(e) $x - 5 = \frac{1}{x-1}$

Calculator Free

4. [12 marks: 2, 2, 2, 3, 3]

Solve for real values of x:

(a) $\sqrt{x+1} = 5$

(b) $\sqrt{x^2+16} = 5$

(c) $\sqrt[3]{2x+3} = 2$

(d) $\sqrt{5-4x} = x$

(e) $x = \sqrt{4x-3}$

---

5. [9 marks: 2, 2, 2, 3]

Solve simultaneously for x and y (where possible):

(a) $x + y = 10$, $x = -4$

(b) $x + y = 10$, $x - y = 8$

Calculator Free

5. (c) $2x + y = 10, 4x + 2y = 8$

(d) $2x + 3y = 4, 3x + y = -1$

---

6. [10 marks: 2, 2, 2, 2, 2]

Solve simultaneously for x and y where x and y are both integers:

(a) $x^2 + y^2 = 10, x = -1$

(b) $(x - 1)^2 + (y + 2)^2 = 13, y = 1$

(c) $x^2 + y^2 = 2, x + y = 0$

(d) $x^2 + y^2 = 5, x + y = 3$

(e) $x^2 + y^2 = 41, x + y = 9$

Calculator Assumed

7. [9 marks: 1, 1, 2, 2, 3]

Solve for x in exact form.

(a) $\frac{5x}{3}+\frac{1}{7} = \frac{3}{4}-\frac{2x}{5}$

(b) $\frac{(x+2)}{4}+3 = \frac{1}{3}-\frac{5(x-7)}{2}$

(c) $2x^2 - 15 = 0$

(d) $x^2 + 2x - 5 = 0$

(e) $x^3 + 2x^2 - 11x - 12 = 0$

---

8. [8 marks: 1, 1, 1, 2, 3]

Solve for x correct to 4 decimal places.

(a) $\frac{(2x-1)}{7}-x = \frac{2x}{3}-\frac{3(x+5)}{4}$

(b) $\sqrt[3]{7x-2} = 5$

(c) $\sqrt{3+2x} = x$

Calculator Assumed

8. (d) $(x+5)^2 - 11 = 0$

(e) $x^3 + 3x^2 = 4x + 10$

---

9. [14 marks: 2 each]

Solve for x and y to the specified accuracy:

(a) $y = \frac{1}{x}$ and $y = x - 3$ (Two decimal places)

(b) $y = -x + 5$ and $y = \frac{5}{2x-1}$ (Two decimal places)

(c) $5x + 3y = 10,\ 7x - 8y = -12$ (Exact Answers)

(d) $1.2x - 3.5y = -0.9,\ 6.1x + 3.6y = 4.2$ (Four decimal places)

(e) $x^2 + y^2 = 1,\ x + y = 0$ (Exact Answers)

(f) $x^2 + y^2 = 5,\ x - y = 2$ (Two decimal places)

(g) $(x-1)^2 + (y+1)^2 = 2,\ x + y = 1$ (Two decimal places)

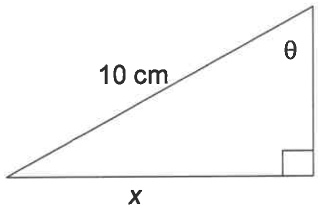
12 Right Triangle Trigonometry

Calculator Free

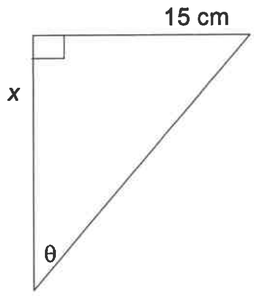
1. [4 marks: 2, 2]

Given that $\sin\theta = 0.6$, $\cos\theta = 0.8$, $\tan\theta = 0.75$, find x in the following triangles:

(a)

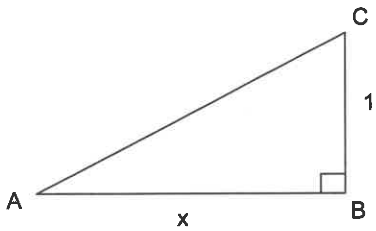


(b)



2. [5 marks: 1, 1, 3]

For ΔABC as shown, find :



(a) AC in terms of x.

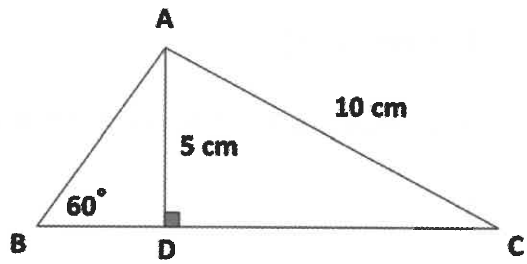
(b) $\cos \angle CAB$ in terms of x.

(d) the exact value of x if $\cos \angle CAB = \frac{\sqrt{7}}{7}$.

Calculator Free

3. [7 marks: 2, 2, 3]

In ΔABC, $\angle ABD = 60^\circ$, AD = 5 cm and AC = 10 cm. AD is perpendicular to BC.



(a) Find BD.

(b) Find $\angle DAC$.

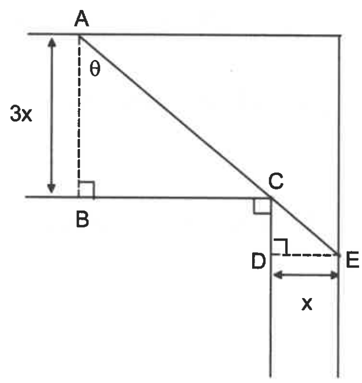
(c) Find BC.

---

4. [4 marks]

Triangles ABC and CDE are right-angled triangles with BC parallel to DE and AB parallel to CD. DE = x and AB = $3x$.

Prove that $AE = x\left[\dfrac{1}{\sin\theta} + \dfrac{3}{\cos\theta}\right]$.



Calculator Assumed

5. [9 marks: 1, 2, 6]

A light aircraft flies horizontally at a speed of 120 kmh^{-1}. During the flight, the pilot noted that it took the plane 30 seconds to fly from being at an angle of depression of 40° to a farmhouse to being directly overhead.

(a) Find the horizontal distance between the aircraft and the farmhouse at the instant the angle of depression to the farmhouse is 40°.

(b) Find the altitude of the aircraft.

(c) Immediately after passing the farmhouse, the aircraft climbs at an angle of 15° to the horizon for 2 minutes. Find the angle of elevation of the aircraft from the farmhouse at the end of the two minutes.

Calculator Assumed

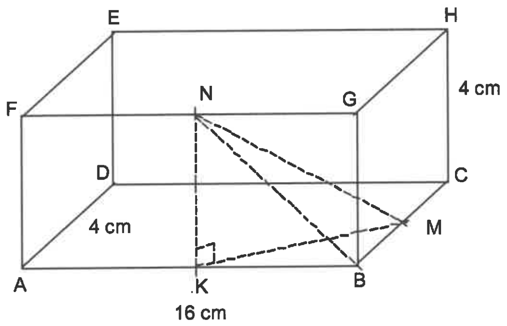
6. [4 marks]

A ball is caught between the branches of a tree. The angle of elevation of the ball from a point A on the ground is $40°$. From a second point B on the ground, 4 metres closer to the foot of the tree than A, the angle of elevation of the ball is $45°$. Assume that A, B and the ball are in the same vertical plane. Find the vertical distance between the ball and the ground.

Calculator Assumed

7. [10 marks: 2, 2, 2, 4]

In the rectangular box shown, M and N are the midpoints of BC and FG respectively. AB = 16 cm, AD = 4 cm and HC = 4 cm. Let K be the midpoint of AB. Find:



(a) the exact length of MK.

(b) the exact length of MN.

(c) the angle between MN and the plane ABCD.

(d) the acute angle between the planes EFBC and ADHG.

13 Non-Right Triangle Trigonometry

Calculator Free

1. [4 marks: 2, 2]

Given that $\sin 45^\circ = \dfrac{\sqrt{2}}{2}$ and $\sin 60^\circ = \dfrac{\sqrt{3}}{2}$:

(a) find x if $\dfrac{x}{\sin 60^\circ} = \dfrac{15}{\sin 45^\circ}$.

(b) find $\sin\theta$ if $\dfrac{10}{\sin\theta} = \dfrac{15}{\sin 45^\circ}$.

---

2. [8 marks: 2, 2, 3]

(a) Find $x > 0$ if $x^2 = (\sqrt{2})^2 + 3^2 - 2 \times \sqrt{2} \times 3 \times \cos\theta$ where $\cos\theta = \dfrac{1}{\sqrt{2}}$.

(b) Find $\cos\theta$ in exact form if $8^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta$.

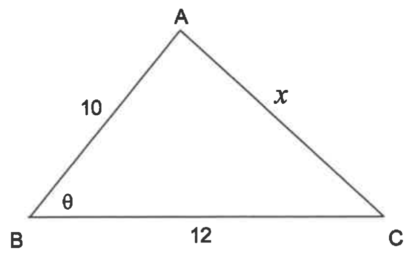
(c) Find $x > 0$ if $(\sqrt{3})^2 = 1^2 + x^2 - 2x\cos 60^\circ$.

Calculator Free

3. [6 marks: 2, 2, 2]

In triangle ABC drawn below, find:

(a) the exact value of x if $\cos\theta = \dfrac{1}{2}$.



(b) $\cos\theta$ in exact form if $x = 12$.

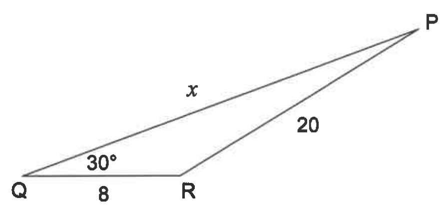
(c) Find the area of ΔABC if $\sin\theta = \dfrac{\sqrt{2}}{2}$.

---

4. [5 marks: 2, 3]

In the accompanying ΔPQR:

(a) find the exact value of sin ∠QPR.



(b) show that the length of the side PQ satisfies the equation $x^2 - 8\sqrt{3}\,x - 336 = 0$.

Calculator Free

5. [7 marks: 3, 1, 3]

P, Q and R are three spots on a large level farm land. Q is located 1 km from P along bearing 150°. R is located 2 km from Q along bearing 210°.

(a) Draw a clearly labelled diagram indicating relative positions of P, Q and R. State all relevant angles and distances.

(b) Find the bearing of P from Q.

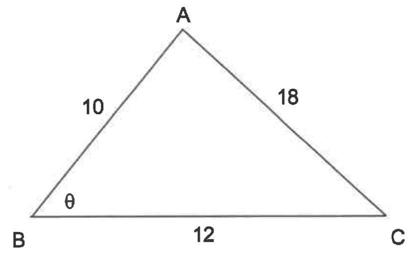
(c) Find in exact form the distance between P and R.

Calculator Assumed

6. [3 marks: 2, 1]

In triangle ABC shown, find:

(a) the exact value of $\cos \theta$.



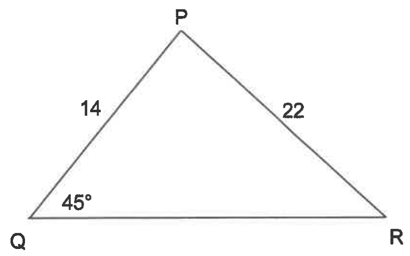
(b) θ giving your answer(s) to the nearest 0.1 of a degree.

---

7. [3 marks: 2, 1]

In triangle PQR shown, find:

(a) the exact value of $\sin \angle PRQ$.



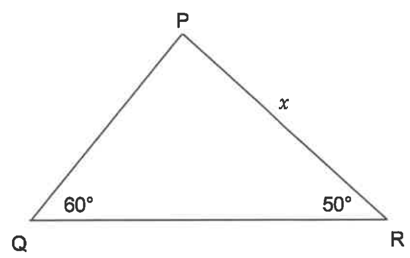
(b) $\angle PRQ$ giving your answer(s) to the nearest 0.1 of a degree.

---

8. [4 marks: 2, 2]

In triangle PQR shown, find:

(a) the length of PQ in terms of x.

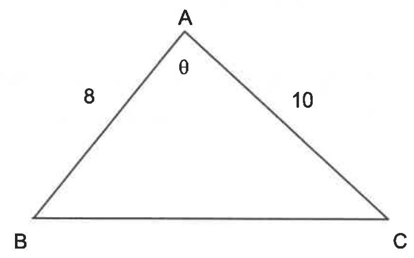


(b) the length of QR in terms of x.

Calculator Assumed

9. [4 marks: 2, 2]

In triangle ABC shown, find:



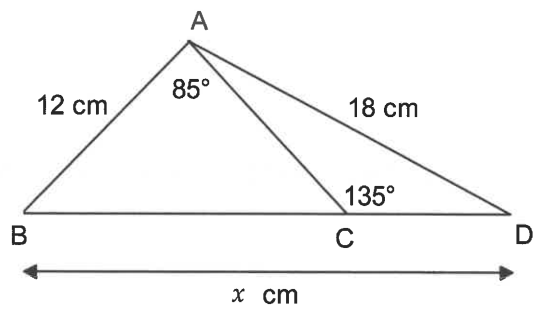
(a) the length of BC in terms of θ.

(b) the size of ∠ACB if θ = 80°.

---

10. [11 marks: 3, 3, 3, 2]

In the accompanying diagram:



(a) find BC to 4 decimal places.

(b) find AC to 4 decimal places.

(c) find CD to 4 decimal places.

(d) hence, find x to 2 decimal places.

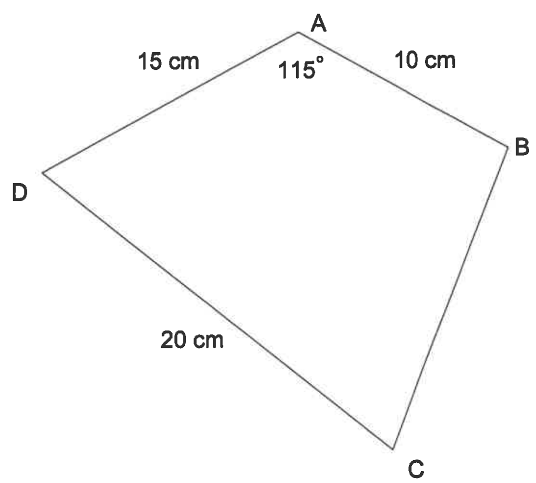
Calculator Assumed

11. [4 marks]

Use the cosine rule to prove that it is impossible to have a triangle with sides measuring 10 cm, 12 cm and 26 cm.

12. [5 marks]

Find the area of quadrilateral ABCD given that $\angle BDC = 40°$.



Calculator Assumed

13. [5 marks]

An aeroplane flying at a constant altitude (height) of h km is sighted at an angle of elevation of 40°. A few minutes later the plane had a flown a further 5 km and is sighted at an angle of elevation 30°. Find h.

14. [9 marks: 6, 3]

From where James is, the referee is 10 metres away on bearing 50°. From where Chris is, the referee is on bearing 290°. James is 30 metres from Chris.

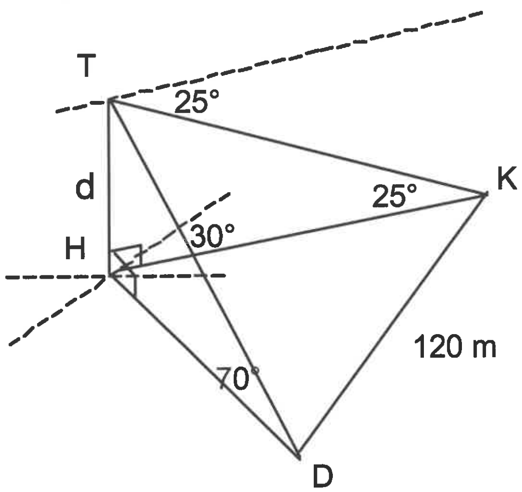
(a) Find the distance between Chris and the referee.

(b) Find the bearing of Chris from James.

Calculator Assumed

15. [5 marks]

From the top of an observation tower of height d metres, a kangaroo is spotted on the ground on an angle of depression of 25° along bearing 030°. A dingo is also spotted on the ground on an angle of depression of 70° along bearing 110°. The dingo is estimated to be 120 metres away from the kangaroo.



Find the height of the observation tower.

14 Arcs, Sectors & Segments

Calculator Free

1. [4 marks]

Complete the following table.

Angle in Degrees	Angle in Radians (exact values)
0	
30	
45	
60	
90	
120	
150	
180	

---

2. [4 marks: 2, 2]

A circular sector is removed from a circle of radius 4 cm.

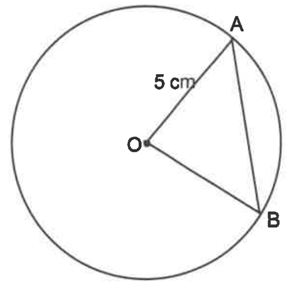
(a) Find the angle of this circular sector (in radians) if the area of this sector is 2π cm^2.

(b) Find the angle of this circular sector (in radians) if the perimeter of the sector is 16 cm.

Calculator Free

3. [8 marks: 2, 2, 4]

In the circle of radius 5 cm with centre O drawn below, $\angle AOB = 60^\circ$.



(a) Find the *exact* area of triangle OAB

(b) Find the *exact* area of the minor segment formed by the chord AB.

(c) Find the exact perimeter of the minor segment formed by the chord AB.

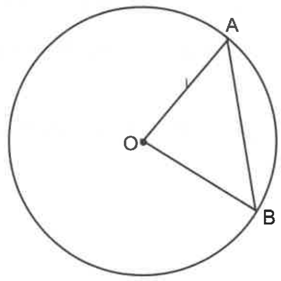
Calculator Assumed

4. [8 marks: 2, 3 3]

In the circle of radius 2π cm with centre O

$\angle AOB = \dfrac{\pi}{3}$.

Find the *exact* (as a multiple or fraction of π) :



(a) area of triangle OAB

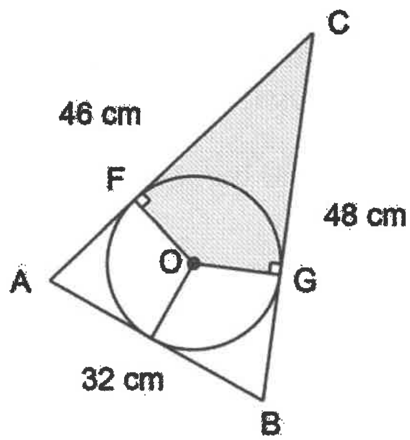
(b) perimeter of the <u>*major*</u> segment formed by the chord AB.

(c) area of the major segment formed by the chord AB.

Calculator Assumed

5. [11 marks: 3, 2, 2, 4]

The accompanying diagram shows a circle of radius 11 cm enclosed within triangle ABC. The circle touches all three sides of the triangle.



(a) Find the size of ∠ACB.
Give your answer to the nearest degree.

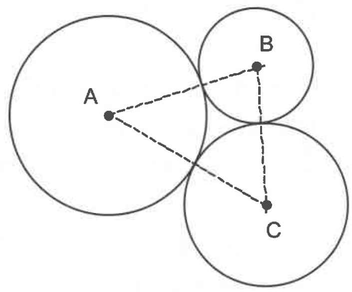
(b) Hence, find the obtuse ∠FOG. Give your answer to the nearest degree.

(c) Find the area of the minor sector FOG.

(d) Find the area of the shaded region.
Show clearly how you obtained your answer.

Calculator Assumed

6. [10 marks: 5, 5]



Three circles of radii 5 cm, 3.5 cm and 2 cm are drawn touching each other as shown in the accompanying diagram.

(a) Find the size of all angles within triangle ABC.

(b) Find the area of the region trapped by the three circles.

15 Trigonometric Equations I

Calculator Free

1. [0 marks]

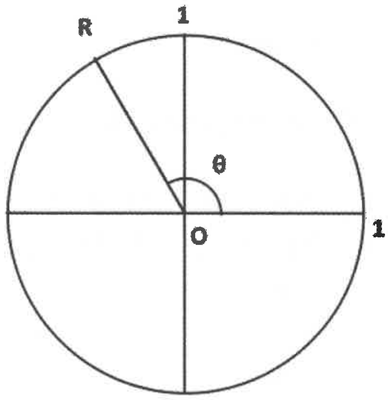
Complete the following table. Give answers in exact form.

Angle θ in degrees	Angle θ in radians	sin θ	cos θ	tan θ
0^{o}				
30^{o}				
45^{o}				
60^{o}				
90^{o}				
120^{o}				
135^{o}				
150^{o}				
180^{o}				
210^{o}				
225^{o}				
240^{o}				
270^{o}				
300^{o}				
315^{o}				
330^{o}				
360^{o}				

Calculator Free

2. [9 marks: 1, 3, 1, 2, 2]

The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates $(-\frac{1}{3}, k)$.



(a) Find cos θ.

(b) Find the two possible values of k.

(b) Hence, find:
(i) sin θ.

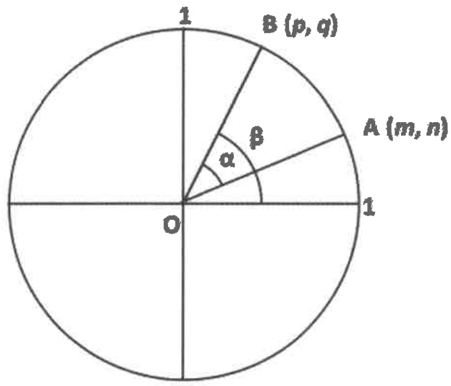
(ii) tan θ.

(iii) sin (180° – θ).

Calculator Free

3. [10 marks: 1, 1, 2, 3, 3]

The angle β is defined by the ray OB. The angle α is the angle trapped between the rays OA and OB. O is the centre of the unit circle. A is a point on the unit circle with coordinates (m, n). B is a point on the unit circle with coordinates (p, q). m, n, p and q are all positive numbers.



(a) Find cos (180° – β).

(b) Find sin (–β).

(c) Find sin (90° – β).

(d) Find cos (90° + β).

(e) Find tan (β – α).

Calculator Free

4. [5 marks: 2, 3]

Find the equation of the line:

(a) passing through the origin and inclined at an angle of 30^o with the positive x-axis.

(b) passing through the point $(\sqrt{3}, 4)$ and inclined at an angle of 60^o with the positive x-axis.

---

5. [6 marks: 3, 3]

Given that $\tan 15^o = -\sqrt{3} + 2$, find the equation of the line:

(a) passing through the origin, inclined at an angle of 165^o with the positive x-axis.

(b) passing through the origin, inclined at an angle of 75^o with the positive x-axis.

Calculator Free

6. [18 marks: 3, 3, 3, 5, 4]

Solve for θ within the given domain:

(a) $\sin\theta = \dfrac{\sqrt{3}}{2}$ where $0^\circ \le \theta \le 360^\circ$

(b) $\cos\theta = -\dfrac{\sqrt{2}}{2}$ where $0 \le \theta \le 2\pi$

(c) $\tan\theta = \sqrt{3}$ where $-\pi < \theta \le \pi$

(d) $\sin(2\theta) = -0.5$ where $-\pi \le \theta \le \pi$.

(e) $\sin\theta = \cos\theta$ where $-180^\circ < \theta \le 180^\circ$

Calculator Free

7. [17 marks: 4, 4, 4, 5]

(a) Given that $\cos 66.4^\circ = 0.4$, solve for θ in $\cos(\theta + 30^\circ) = 0.4$ where $0^\circ \le \theta \le 360^\circ$.

(b) Given that $\tan 26.6^\circ = 0.5$, solve for θ in $1 - 2\tan(\theta + 6.6^\circ) = 0$ where $0 \le \theta \le 360^\circ$.

(c) $(\sin\theta - 2)(2\sin\theta - 1) = 0$ where $0^\circ \le \theta \le 360^\circ$

(d) $2\cos^2\theta + 3\cos\theta - 2 = 0$ where $0 \le \theta \le 2\pi$

Calculator Assumed

8. [4 marks: 2, 2]

(a) Find the angle the line with equation $y = 2x + 5$ makes with the positive x-axis.

(b) Line L has equation $3x + 4y = 12$.
Find the angle this line makes with the positive x-axis.

---

9. [5 marks]

Lines L1 and L2 have equations $x + y = 10$ and $y = \frac{4x}{5} - 3$ respectively.

Find the acute angle between these two lines.

16 Trigonometric Graphs

Calculator Free

1. [6 marks]

Complete the following table.

Function	Period	Amplitude (where applicable)
$y = 2\sin(2x^\circ)$		
$y = -4\cos(\frac{x}{2} + 30^\circ)$		
$v = 10\tan(3t + \pi)$		
$Q = 5\sin(\frac{\pi}{2} - t)$		
$y = \frac{\sqrt{2}}{2}\cos(\pi t) + 100$		
$T = 5 - \sin(\frac{\pi}{4} - \theta)$		

2. [5 marks]

Complete the table below.

Function	Minimum value of function	Maximum value of function
$y = 3\sin t$		
$y = 20\cos(\frac{2x}{3} - 45^\circ)$		
$v = 5\tan\theta$		
$M = 2\sin(\frac{\pi}{2} - 3t) + 4$		
$y = 5 - \cos(2\pi t)$		

Calculator Free

3. [8 marks: 4, 4]

A trigonometric function has equation $y = -4 \sin (2x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$. Find:

(a) the maximum value for y and the corresponding value(s) for x.

(b) the minimum value for y and the corresponding values for x.

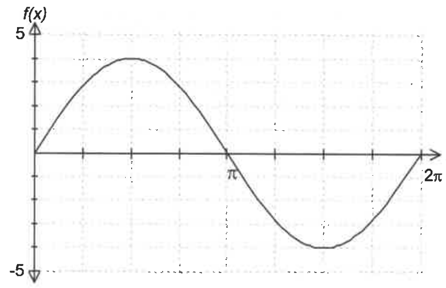
4. [4 marks]

A trigonometric function has equation $P = a \cos (bt + \frac{\pi}{4})$. Find the values of a and b given that P has a maximum value of 4 and a period of 4.

Calculator Assumed

5. [3 marks]

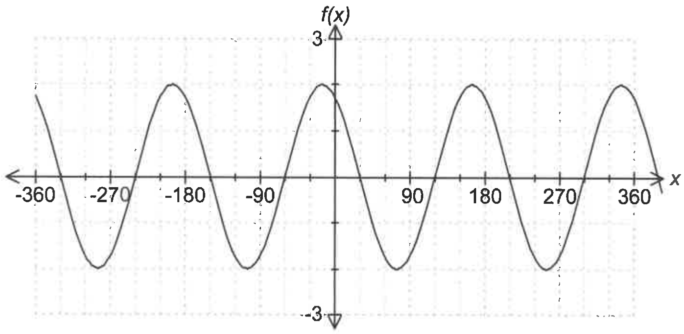
The accompanying diagram shows the graph of a trigonometric function. State the amplitude and period of the function. Hence, give the equation of this function.



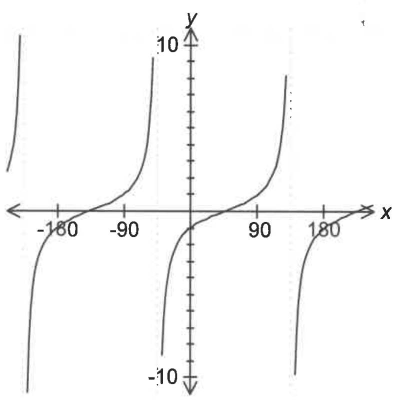
6. [7 marks: 4, 3]

Find the equation of the following trigonometric functions:

(a)



(b)



Calculator Assumed

7. [11 marks: 1, 1, 2, 2, 5]

The body temperature θ (Celsius) of a reptile in summer at time t hours after midnight is given by $\theta = 15 - 5\sin(\frac{\pi t}{12})$.

(a) State the period for θ.

(b) What is the range of body temperature experienced by the reptile?

(c) Find the minimum body temperature of the reptile and state when this first occurs after midnight.

(d) Find the maximum body temperature of the reptile and state when this first occurs after midnight.

(e) Use an algebraic method to find the first time when the temperature of the reptile is 16° Celsius.

17 Trigonometric Identities (Add/Sub Formulae)

Calculator Free

1. [13 marks: 4, 4, 5]

Use an appropriate trigonometric identity to find the exact value of :

(a) $\sin 75^{\circ}$

(b) $\cos 165^{\circ}$

(c) $\tan \dfrac{7\pi}{12}$

Calculator Free

2. [10 marks: 2, 1, 3, 4]

Given that $\sin A = \dfrac{4}{5}$ and $0 < A < \dfrac{\pi}{2}$, find the exact value of:

(a) $\cos A$

(b) $\tan A$

(c) $\sin\left(\dfrac{\pi}{2} + A\right)$

(d) $\cos\left(\dfrac{\pi}{4} - A\right)$

Calculator Free

3. [13 marks: 2, 2, 3, 3, 3]

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{4}$, where A and B are acute, find the exact value of:

(a) $\cos A$

(b) $\sin B$

(c) $\sin(A + B)$

(d) $\cos(A - B)$

(e) $\tan(A + B)$

Calculator Assumed

4. [13 marks: 2, 2, 3, 3, 3]

Given that $\sin P = \dfrac{5}{13}$ and $\cos Q = -\dfrac{15}{17}$, where P and Q are each obtuse angles, find the exact value of:

(a) $\cos P$

(b) $\sin Q$

(c) $\sin(P - Q)$

(d) $\cos(P + Q)$

(e) $\tan(P - Q)$

18 Trigonometric Equations II

Calculator Free

1. [13 marks: 3, 5, 5]

Solve for x within the given domain:

(a) $\cos x + \sqrt{3}\sin x = 0$ $\quad 0 \le x \le 360^\circ$:

(b) $2\sin^2 x - 3\sin x - 2 = 0$ $\quad$ for $0 \le x \le 360^\circ$

(c) $\cos x - \dfrac{3}{\cos x} - 2 = 0$ $\quad$ for $0 \le x \le 2\pi$

Calculator Free

2. [16 marks: 5, 5, 6]

Solve for θ within the given domain:

(a) $\cos(\theta + 30^\circ) = \sin\theta$ for $0 \le \theta \le 360^\circ$

(b) $\sin(\theta + \frac{\pi}{4}) = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$

(c) $\sin(\theta - \frac{\pi}{4}) = -\sqrt{2}\cos(\theta + \frac{\pi}{6})$ for $0 \le \theta \le 2\pi$

Calculator Free

3. [14 marks: 2, 5, 7]

(a) Use the formula for sin (A + B) to show that sin 2A = 2 sin A cos A.

(b) Use the formula in (a) to solve for x in $\cos x + \sin 2x = 0$ for $0 \leq x \leq 360^{\circ}$.

(c) Use the formula in (a) to solve for x in $\sin 2x - \sin x = 0$ for $0 < \theta < 2\pi$.

19 Sets

Calculator Assumed

1. [6 marks: 1, 1, 2, 2]

Given that $\mathbf{U} = \{ x \mid 50 \le x \le 70, x \text{ is an integer}\}$,
$\mathbf{A} = \{51, 53, 65, 68\}$, $\mathbf{B} = \{62, 64, 65, 66\}$ and $\mathbf{C} = \{51, 53, 66, 70\}$.

(a) Find $|\mathbf{U}|$.

(b) Find $\mathbf{A} \cup \mathbf{B}$.

(c) Find $n(\mathbf{C} \cap \overline{\mathbf{B}})$.

(d) | Find $\overline{\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}}$ |.

---

2. [7 marks: 1, 2, 2, 2]

Given that $\mathbf{U} = \{ x \mid 0 \le x \le 20, x \text{ is an integer}\}$,
$\mathbf{P} = \{ x \mid 5 \le x \le 14\}$, $\mathbf{Q} = \{ x \mid 3 \le x \le 9\}$ and $\mathbf{R} = \{2, 4, 6\}$.

(a) Is $2 \in \mathbf{R}$?

(b) Is $\mathbf{Q} \subset \mathbf{P}$? Justify your answer.

(c) Find $n(\mathbf{Q}')$.

(d) Find $|\mathbf{U} \cap \mathbf{P}|$.

Calculator Assumed

3. [6 marks: 1, 1, 2, 2]

Given that $\mathbf{U} = \{x \mid -10 \le x \le 10, x \text{ is an integer}\}$,
$\mathbf{A} = \{x \mid -10 \le x \le -1\}$, $\mathbf{B} = \{x \mid 0 \le x \le 10\}$ and $\mathbf{C} = \{2, 3, 5, 7\}$.

(a) Find $\mathbf{A} \cap \mathbf{B}$.

(b) Find $\mathbf{A} \cup \mathbf{B}$.

(c) Find $n(\mathbf{B} \cap \overline{\mathbf{A}})$.

(d) Find $|(\mathbf{A} \cup \mathbf{B}) \cap \overline{\mathbf{C}}|$.

---

4. [8 marks: 2, 2, 2, 2]

Given that $\mathbf{U} = \{x \mid 1 \le x \le 20, x \text{ is an integer}\}$, $\mathbf{A} = \{x \mid x \text{ is a prime number}\}$,
$\mathbf{B} = \{x \mid x \text{ is a square number}\}$ and $\mathbf{C} = \{x \mid x \text{ is a multiple of } 3\}$.

(a) Find $\mathbf{B} \cap \mathbf{C}$.

(b) Find $\mathbf{A} \cup \mathbf{B}$.

(c) Find $|\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})|$.

(d) Find $|(\mathbf{A} \cup \mathbf{B}) \cap \mathbf{C})|$.

Calculator Assumed

5. [9 marks: 2, 2, 3, 2]

Given that **U** = $\{x \mid 0 \le x \le 20, x \text{ is an integer}\}$, **A** = $\{x \mid x \text{ is a factor of } 24\}$, **B** = $\{x \mid x \text{ is a prime number}\}$ and **C** = $\{x \mid x \text{ is a triangular number}\}$.

(a) Find $\mathbf{B} \cap \mathbf{C}$

(b) Find $\mathbf{C} \cup \mathbf{B}$

(c) $n(\mathbf{A} \cap \overline{\mathbf{B}})$

(d) An element is chosen at random from **U**. Find the probability that this element is from set **B**, given that it is from set **C**.

---

6. [5 marks: 1, 1, 1, 2]

Given that **A** = $\{1, 2, 3\}$, **B** = $\{0, 1, 2\}$ and **C** = $\{(x, y) \mid x \in A, y \in B\}$.

(a) Find $\mathbf{A} \cap \mathbf{B}$

(b) Find $\mathbf{A} \cup \mathbf{B}$

(c) Is $(1, 2) \in C$?

(d) $|\mathbf{C}|$

20 Combinations

Calculator Free

1. [10 marks: 1, 2, 2, 2, 3]

Evaluate each of the following:

(a) $5!$

(b) $\binom{10}{5}$

(c) $\binom{10}{5} \times 5!$

(d) $\binom{40}{2}$

(e) $\binom{60}{58}$

Calculator Free

2. [7 marks: 1, 2, 2, 2]

Determine the value of integer r (where $r \geq 0$) in each of the following equations:

(a) $\binom{12}{8} = \binom{12}{r}$

(b) $\binom{30}{r} = \binom{30}{r+4}$

(c) $\binom{25}{2r} - \binom{25}{r-2} = 0$

(d) $\binom{r}{0} + \binom{r}{1} + \binom{r}{2} + \binom{r}{3} + \binom{r}{4} = 2^r$

---

3. [4 marks: 1, 3]

Expand completely each of the following:

(a) $(1 + x)^5$

(b) $(2 - x)^4$

Calculator Free

4. [13 marks: 1, 4, 5, 3]

Consider the expansion for $\left(x^2 - \frac{2}{x}\right)^{12}$ in *descending* powers of x.

(a) How many terms are there in this expansion?

(b) Find the third term in this expansion.

(c) Find a mathematical expression for the coefficient of the term in $\frac{1}{x^{12}}$.

(d) Find a mathematical expression for the term independent of x.

Calculator Free

5. [5 marks: 1, 2, 2]

Amy has a collection of 18 fluoro pens in her pink box and 24 fluoro pens in her blue box. Write mathematical expressions for the number of ways Amy can pick:

(a) three pens from her pink box.

(b) three pens from the pink box and four pens from the blue box.

(c) a dozen pens from both boxes.

---

6. [10 marks: 1, 2, 2, 3, 2]

A committee of 9 people is to be selected from 10 Labor, 8 Liberal and 5 Green politicians. Write mathematical expressions for the number of different ways the committee can be selected if:

(a) there are no restrictions.

(b) all three political parties are equally represented.

(c) there are no Greens.

(d) the Liberal representatives are in the majority.

(e) the Labor husband and wife pair, Alex and Alice, cannot be in the same committee.

Calculator Assumed

7. [19 marks: 1, 3, 3, 4, 4, 4]

Consider the digits 0 to 9 inclusive and all the letters of the alphabet. Ten characters consisting of digits and letters are chosen. Determine the number of ways of choosing:

(a) all the even numbers and all the vowels.

(b) any six digits and any four letters.

(c) exactly four vowels.

(d) at least four odd digits.

(e) four vowels and four odd digits.

(f) four vowels or four odd digits.

21 Probability I

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

Let the universal set U be the set of all integers between 1 and 20 inclusive.
Let M: $\{x \mid x \text{ is a multiple of } 6\}$ and F: $\{x \mid x \text{ is a factor of } 72\}$.

(a) List all the elements in F.

Use a Venn Diagram or a two-way table to the answer the following questions.

(b) How many multiples of 6 are not factors of 72?

(c) How many integers are either multiples of 6 or are factors of 72?

(d) Find the probability that a randomly chosen integer:
(i) is a neither a multiple of 6 nor a factor of 72.

(ii) is a multiple of 6 given that it is a factor of 72.

Calculator Assumed

2. [8 marks: 2, 2, 2, 2]

The Mathematics department at a school conducted a random survey involving 200 students.

- 38 students did not have any calculator (scientific or CAS) with them
- 142 students had a CAS calculator with them
- 52 students had a scientific calculator with them

Use a Venn Diagram or a two-way table to the answer the following questions.

(a) Find the probability that a student chosen at random had only a CAS calculator.

(b) Find the probability that a student chosen at random had both a CAS calculator as well as a scientific calculator.

(c) Find the probability that a student chosen at random had either a CAS calculator or a scientific calculator.

(d) Find the probability that a student selected from those who had at least one type of calculator had both types of calculator.

Calculator Assumed

3. [8 marks: 2, 2, 2, 2]

In a group of 40 students, there are 10 boys who are colour vision impaired (CVA) and 15 girls who are not colour vision impaired. There are as many boys who are not colour vision impaired as there are boys who are colour vision impaired.

Use a Venn Diagram or a two-way table to the answer the following questions.

(a) How many girls were colour vision impaired?

(b) A student is randomly chosen from this group. Find the probability that this student is a girl.

(c) A student is randomly chosen from this group. Find the probability that this student is either a girl or is colour vision impaired.

(d) A student is randomly chosen from this group. Given that this student is either a girl or is colour vision impaired, find the probability that this student is colour vision impaired.

Calculator Assumed

4. [10 marks: 4; 1, 1, 2, 2]

Box A contains two red balls and a white ball. Box B contains one white ball and a red ball. A coin is tossed. If a tail appears, 2 balls are drawn without replacement from box B. If a head appears, 2 balls are drawn without replacement from box A.

(a) Use a tree diagram to display all the possible outcomes.

(b) If each outcome is equally likely, find the probability that:

(i) exactly two red balls are chosen.

(ii) exactly one white ball is chosen.

(iii) exactly two red balls are chosen given that box A was chosen.

(iv) box A was chosen given that exactly one white ball was chosen.

Calculator Assumed

5. [10 marks: 4, 6]

Allie's Café offers the following menu:

Appetisers: *Prawn Cocktail or Bruschetta*
Main Meal: *Fish or Chicken or Steak or Lamb*
Dessert: *Pavlova or Cheesecake or Apple Pie with Ice Cream*

Jenny orders an appetiser, one item for the main meal and one dessert. Jenny will not have Steak with Pavlova and must have Prawn Cocktail with Fish and Fish only with Prawn Cocktail.

(a) Display Jenny's possible appetiser/main meal/dessert combinations in a clearly labelled tree diagram.

(b) If each of Jenny's possible appetiser/main meal/dessert combination are equally likely, find the probability that Jenny:

(i) chose chicken as a main meal.

(ii) did not choose cheesecake as a dessert.

(iii) chose cheesecake given that she chose chicken.

(iv) chose chicken given that she chose cheesecake.

Calculator Assumed

6. [10 marks: 1, 3, 3, 3]

A red box has four books and a blue box has eight books. All books are different. A total of five books are chosen from these two boxes.

(a) In how many ways can this be done?

(b) What is the probability that all the books from the red box are chosen?

(c) What is the probability that at least one of the books chosen is from the red box?

(d) What is the probability that more books from the red box are chosen?

Calculator Assumed

7. [15 marks: 3, 3, 3, 3, 3]

Consider the digits 0 to 9 inclusive and all the letters of the English Roman alphabet. Twelve characters consisting of digits and letters are chosen.

(a) What is the probability that all the characters chosen are letters?

(b) What is the probability that all the digits are chosen?

(c) Given that all the characters chosen are letters, what is the probability that all the vowels are chosen?

(d) What is the probability that no vowels or even digits were chosen?

(e) What is the probability that at least one vowel or even digit was chosen?

Calculator Assumed

8. [11 marks: 1, 1, 1, 2, 3, 3]

Last year, Malcolm was late to school on average, 5 days out of 100 days. Write mathematical expressions (but do not evaluate) for the probability that in a school week of 5 days, Malcolm is:

(a) late only on the first day.

(b) late on the first three days.

(c) late only on the first three days.

(d) late only on exactly three days.

(e) late on at least three days.

(f) late only the first and the fifth day given that he was late on exactly two days in the school week.

Calculator Assumed

9. [9 marks: 1, 2, 3, 3] [TISC]

Zico practices kicking a soccer ball from the penalty spot. From previous practices, on average, he scores 70 goals from 100 attempts.

(a) Find the probability that Zico's first two kicks do not score goals.

(b) Find the probability that Zico's first kick scores a goal but the next two kicks do not score goals.

If Zico has 10 kicks of the ball from the penalty spot, find the probability that,

(c) he scores exactly 5 goals.

(d) he scores goals only on the first, fifth, seventh and ninth kick.

Calculator Assumed

10. [11 marks: 2, 3, 3, 3]

The *Collett Boat Company* has a fleet of three boats. From Company records for the last two years, the *Jupiter* is chosen by 55% of customers, the *Venus* by 28% of customers and the *Mars* by the remaining customers. The probabilities that each boat breaks down during a two-hour trip are *Jupiter* 0.2; *Venus* 0.15; *Mars* 0.3.

(a) If all three boats are out on hire for a two-hour trip, find the probability that:
(i) none breaks down.

(ii) the *Mars* and one other boat in this fleet breaks down.

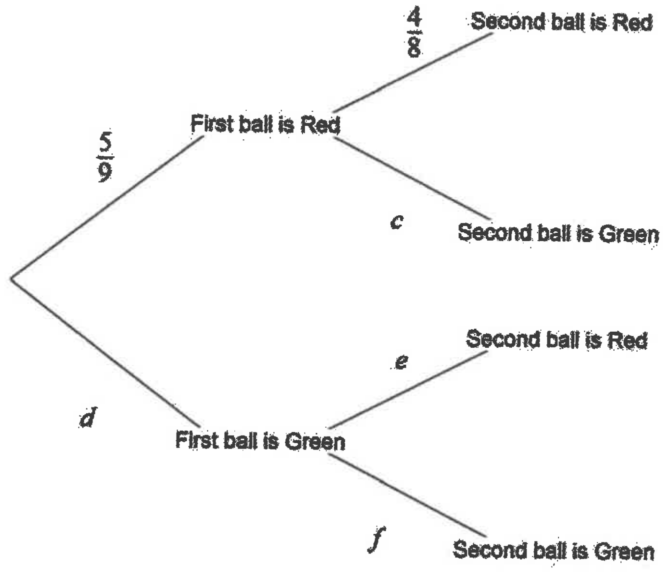
(b) Only one boat is out on hire for a two-hour trip. What is the probability that it will break down.

(c) News comes through that the one boat out on hire has broken down. What is the probability that it is the *Jupiter*?

Calculator Assumed

11. [9 marks: 4, 2, 3]

A box has five red balls and four green balls. Two balls are drawn without replacement from this box. The tree diagram indicates the associated outcomes and the corresponding probabilities.



(a) State the probability values c, d, e and f.

(b) Find the probability that both balls are red.

(c) Find the probability that both balls are of the same colour. Show clearly how you obtained your answer.

Calculator Assumed

12. [10 marks: 2, 4, 4] [TISC]

Chin either drives to work or takes a train to work. The probability that he is on time for work is 0.86. The probability that he is late for work given that he drives to work is 0.3. The probability that he is on time for work given that he takes a train is 0.9.

(a) Find the probability that he is on time for work given that he drives to work.

(b) Find the probability that he drives to work.

(c) Given that he is late for work, what is the probability that he took the train to work.

Calculator Assumed

13. [9 marks: 1, 5, 3]

At a certain airport, the probability that a plane takes off on time given that weather conditions are fine is 0.9. The probability that a plane takes off on time given that weather conditions are bad is 0.7. The probability of weather conditions being fine or the plane taking off on time is 0.955.

(a) Find the probability that a plane does not take off on time given that weather conditions were bad.

(b) Find the probability of a plane taking off on time in fine weather conditions.

(c) Find the probability of weather conditions being fine given that a plane took off on time.

Calculator Assumed

14. [11 marks: 5, 6]

To be awarded a pass for a course, students are required to first pass a practical examination and then a theory examination. A student is only allowed one failure. A student that fails two examinations automatically fails the course. In all cases, a student is allowed to repeat a failed examination only once. The probability that a student will pass the practical examination on the first attempt is 0.8. The probability that a student will pass the theory examination given that the student has passed the practical examination is 0.9, even if the student failed the practical examination in the first attempt. The probability that a student fails outright (fails the practical examination on the first and second attempt) is 0.01 and the probability that a student fails the course is 0.045.

(a) Find the probability that a randomly chosen student fails the entire course given that the student failed the practical examination on the first attempt.

Calculator Assumed

14. (b) Find the probability that a randomly chosen student passes the course given that the student had to repeat an examination.

22 Probability II

Calculator Free

1. [5 marks: 3, 2]

Given that $P(\overline{A \cup B}) = 0.2$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$;

(a) find $P(B \mid A)$.

(b) determine with reasons if the events A and B are independent.

---

2. [6 marks]

Given that $P(A) = 0.4$, $P(C \mid A) = 0.3$, $P(C \mid \bar{A}) = 0.2$, find $P(A \mid C)$.

Calculator Free

3. [8 marks: 2, 2, 4]

It is known that P(A) = 0.6 and P(B) = 0.3. Find:

(a) $P(B \mid A)$ given that A and B are mutually exclusive.

(b) $P(A \mid B)$ given that A and B are independent.

(c) $P(B \mid A)$ given that $P(A \mid B) = 0.2$.

Calculator Free

4. [8 marks: 4, 2, 2]

Given that $P(A') = \frac{3}{10}$, $P(B|A) = \frac{2}{7}$ and $P(A|B) = \frac{1}{2}$:

(a) find P(B).

(b) find $P(A' \cap B')$.

(c) determine with reasons if A and B are independent.

Calculator Assumed

5. [5 marks: 1, 1, 3]

P(A) = 0.3, P(B′) = 0.4 and P(A ∩ B) = *k*:

(a) find in terms of *k*,

(i) P(A ∩ B′).

(ii) P(A′ ∩ B).

(b) find *k* given that $P(\overline{A \cup B}) = 0.18$.

---

6. [4 marks]

Given that P(A) = 0.5, P(B) = 0.8 and $P(\overline{A} \cap \overline{B}) = 0.05$ and that the events A and B are independent, determine if these results are consistent with the rules of probability. Justify your answer.

Calculator Assumed

7. [10 marks: 1, 2, 4, 3]

Given that $P(A) = p + 0.2$ and $P(B) = p + 0.3$ and $P(A \cap B) = p$, calculate the value of p if:

(a) A and B are mutually exclusive events.

(b) $P(A \cup B) = 0.6$.

(c) A and B are independent events.

(d) $P(A | B) = 0.4$.

Calculator Assumed

8. [9 marks: 4, 3, 2]

A supply company checked its accounts and found that 8% of accounts were in arrears. 60% of the accounts were for sole traders while the other 40% were for companies. Only 2% of accounts were both in arrears and were accounts of sole traders.

(a) Find the probability of an account not being in arrears belonging to a sole trader.

(b) Of accounts in arrears, find the probability of the account belonging to a sole trader.

(c) Is the account status independent of the type of trader? Justify your answer.

Calculator Assumed

9. [7 marks: 1, 2, 2, 2] [TISC]

John drives to work each weekday morning. The route he takes passes through a set of traffic lights where he either has to stop at the lights or move through without stopping.

- If he has to stop at the lights, the probability that he will be late for work is 0.7.
- If he does not have to stop at the lights, the probability that he will be late for work is 0.2.

Overall, the probability that John will be late for work is 0.25.

(a) Find the probability that he will not be late for work if he did not have to stop at the traffic lights.

(b) Find the probability that John has to stop at the traffic lights.

(c) Determine with reasons if John being late is independent of whether he has to stop at the lights.

(d) Find the probability that John had to stop at the lights given that he was late for work.

Calculator Assumed

10. [10 marks: 2, 3, 3, 2] [TISC]

England and Germany play a series of two soccer matches. Each match does not end in a draw. The probability that Germany will win both matches is 0.42 and the probability that Germany will lose both matches is 0.135. The probability that Germany will win the first match is 0.7.

(a) Find the probability that England wins exactly one of the matches.

(b) Find the probability that England win the second match.

(c) Find the probability that Germany lose the first match given that they won the second match.

(d) Determine with reasons if the events that Germany win match one and the event that Germany win match two are independent.

23 Indices

Calculator Free

1. [12 marks: 2, 2, 2, 3, 3]

   Simplify each of the following leaving answers with positive indices:

   (a) $\dfrac{16x^2y^3}{12x}$

   (b) $\left(\dfrac{3x^2}{y}\right)^{-2}$

   (c) $\dfrac{81x^{-4}y^5}{36x^2y^{-2}}$

   (d) $\dfrac{(4x^2y^4z)^{\frac{1}{2}}}{xy^{-1}\sqrt{z}}$

   (e) $\dfrac{x^3y^2}{z^2y^{-1}} \times \dfrac{y\sqrt{z}}{x^{-1}}$

Calculator Free

2. [13 marks: 2, 2, 2, 3, 4]

Simplify each of the following, leaving answers with positive indices:

(a) $\dfrac{5^{n+1}-5^n}{8}$

(b) $\dfrac{5^{n+2}-5^{n+1}}{2(5^{n+1})}$

(c) $\dfrac{7^{n-1}+7^n}{4\times 7^{n-1}}$

(d) $\dfrac{3^{2n}+3^n}{3^{n+1}+3}$

(e) $\dfrac{2^{n+1}+8}{3(2^n)+12}$

Calculator Free

3. [12 marks: 2, 2, 2, 3, 3]

Solve for t.

(a) $3^{2t+1} = 81$

(b) $4^{1-t} = 32$

(c) $5^{2+t} = \frac{1}{125}$

(d) $5^{t} \times 25^{t-1} = 0.04$

(e) $\frac{2^{2t+1}}{2^{1-t}} = 4$

Calculator Free

4. [3 marks]

Solve for x in $(2^x)^2 + 2(2^x) - 8 = 0$.

Hint: Replace 2^x with y.

5. [5 marks]

Solve for x, $3^{2x+1} + 8(3^x) - 3 = 0$.

24 Arithmetic Progressions

Calculator Free

1. [4 marks]

   An arithmetic sequence has first term 10 and common difference 8. State the recursive rule and general rule for this sequence.

---

2. [5 marks: 3, 2]

   The terms of a sequence are defined by $T_{n+1} - T_n + 20 = 0$ with $T_1 = 100$

   (a) Show that this sequence is an arithmetic sequence.

   (b) How many positive terms are there in this sequence?

---

3. [4 marks]

   An arithmetic sequence is described by the rule $T_n = 150 - 4n$, where $n = 1, 2, 3, 4, 5, \ldots\ldots$. Find the first three terms of the sequence. Hence, state the recursive rule for this sequence.

Calculator Free

4. [5 marks: 2, 3]

An arithmetic sequence is described by the rule $T_{n+1} = T_n + 6$ with $T_1 = -96$.

(a) Find the general rule of this sequence in the form $T_n = a + bn$, where a and b are constants and $n = 1, 2, 3, 4, 5, \ldots\ldots$.

(b) How many negative terms are there in this sequence?

---

5. [9 marks: 4, 2, 3]

The sum of the first n terms of an arithmetic progression is given by $S_n = 3n^2 - 21n$. Find:

(a) the first three terms of the sequence.

(b) the recursive rule of the sequence.

(c) the sum of all terms between the 11th term and the 20th term inclusive.

Calculator Free

6. [5 marks]

The eighth term and twelfth term of an arithmetic sequence are 24 and 40 respectively. Find the recursive rule for the sequence.

---

7. [8 marks: 3, 4, 1]

The sixth term of an arithmetic sequence is double its fourth term. The first term of the sequence is 20 and the common difference is d.

(a) Show that $T_4 = 2 \times d$.

(b) Hence, find the general rule for the sequence.

(c) Find the three consecutive terms of this sequence that sum to –60.

Calculator Assumed

8. [6 marks: 3, 1, 2]

An arithmetic sequence has first term –20 and common difference 3.

(a) Find the 20th term of the sequence and the sum of the first 20 terms.

(b) Find the first positive term in the sequence.

(c) Find n so that the sum of the first n terms is positive for the first time.

---

9. [5 marks: 3, 2]

An arithmetic sequence has first term 200 and common difference –10.

(a) Find which term is the first negative term in the sequence.

(b) The sum of the first n terms is 900. Find n.

Calculator Assumed

10. [9 marks: 2, 2, 3, 1, 1]

A special filter is installed to remove airborne particles from a dust-free room. The filter has to be changed every 10 weeks. In the first five weeks of operation, it removes 10 000 particles each week and thereafter its filtering capacity reduces by 500 particles each week.

(a) Write a recursive equation that describes the filtering capacity of the filter for the first five weeks.

(b) Write a recursive equation that describes the filtering capacity of the filter from the sixth week to the tenth week inclusive.

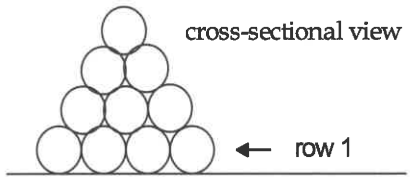
(c) Write in terms of k, an equation that describes the number of particles filtered in week k, if $6 \leq k \leq 10$.

(d) Find the total number of particles filtered in the first 5 weeks.

(e) Find the total number of particles filtered by the end of the 7th week.

Calculator Assumed

11. [11 marks: 2, 2, 3, 4]



Joe runs a hardware store. He stores 25mm diameter cylindrical polyurethane pipes (each of length 10m) used for reticulation systems in the yard and stacks the pipes up in piles similar to the one shown in the accompanying diagram. Each pile has one pipe at the top of the pile.

(a) The bottom row of a pile of pipes has 50 pipes. How many pipes are there in this pile?

(b) Another pile has 18 pipes in its 5th row (row 1 is on the ground). How many pipes are there in this pile?

(c) There are 465 pipes in a pile. How many rows are there in this pile?

Calculator Assumed

11. (d) A new shipment of 100 pipes was delivered. How can the pipes be stacked so that a minimum number of piles are used?

---

12. [6 marks: 3, 1, 2]

Brooke invests \$50 000 in an account that pays simple interest at a rate of 5% per year. The interest is paid at the end of each year and is not added to the principal. Let B(n) be the account balance at the end of n years.

(a) Find the recursive rule and general rule for the account balance after n years.

(b) Find n when the account balance is \$75 000.

(c) Find the minimum number of years required for the balance to exceed \$150 000.

Calculator Assumed

13. [9 marks: 2, 2, 5]

X and Y are two campsites 200 km apart on the Bibbulman Track. Paige is the leader of a group of hikers that start off from Camp X towards Camp Y. This group of hikers walk 5 km on the first day, and thereafter increase their rate by an extra 0.6 km each day until a rate of 8 km a day is achieved. They then walk at this rate for the rest of the trip.

(a) On which day did Paige's group attain the maximum walking rate of 8 km per day?

Jasmine is the leader of a second group of hikers that start off from Camp Y towards Camp X. This group of hikers walk 5 km on the first day, and thereafter increase their rate by an extra 0.5 km each day until a rate of 8 km a day is achieved. They then walk at this rate for the rest of the trip.

(b) On which day did Jasmine's group attain the maximum walking rate of 8 km per day?

(c) If Paige's and Jasmine's group start off on the same day, when and where will the two groups meet along the Bibbulman Track?
Show clearly how you obtained your answer.

Calculator Assumed

14. [9 marks: 3, 6]

A grandfather clock makes as many long chimes as the hour of the day, on the hour, every hour. For example, at 1 pm (or am) it makes one long chime, at 2 pm (or 2 am) it makes two long chimes, ... , at 12 midnight (or 12 noon) it makes twelve long chimes.

(a) How many long chimes would this clock make in a 24 hour day? Show clearly how you obtained your answer.

In addition, this clock makes 1 short chime to mark the passage of the first quarter of any hour, 2 short chimes to mark the passage of the first half-hour of any hour and 3 short chimes to mark the passage of the third quarter of any hour. For example:

- at 1 pm, it will make one long chime.
- at 1.15pm it will make 1 short chime, at 1.30 pm it will make 2 short chimes.
- at 1.45 pm it will make 3 short chimes and at 2 pm it will make two long chimes.

(b) How long after 12 noon, would it take for the clock to have made a total of 105 chimes (long and short)? Justify your answer.

25 Geometric Progressions

Calculator Free

1. [4 marks]

   A geometric sequence has first term 5 and common ratio 2. State the recursive rule and general rule for this sequence.

---

2. [5 marks: 3, 2]

   The terms of a sequence are defined by $\frac{T_{n+1}}{T_n} - \frac{1}{2} = 0$ with $T_1 = 1024$

   (a) Show that this sequence is a geometric sequence.

   (b) How many terms greater than 32 are there in this sequence?

---

3. [4 marks]

   A geometric sequence is described by the rule $T_n = 5 \times 3^n$, where $n = 1, 2, 3, \ldots$. Find the first three terms of the sequence. Hence, state the recursive rule for this sequence.

Calculator Free

4. [5 marks: 3, 2]

A sequence is described by the rule $T_{n+1} = T_n \times 5$ with $T_1 = 2$.

(a) Find the general rule of this sequence in the form $T_n = a \times b^n$, where a and b are constants and $n = 1, 2, 3, 4, 5, \ldots\ldots$.

(b) How many terms less than 1000 are there in this sequence?

---

5. [8 marks: 3, 2, 3]

The sum of the first n terms of a geometric progression is given by $S_n = 4^{n+1} - 4$.

(a) Find the first three terms of the sequence.

(b) Find the general rule of the sequence.

(c) Find a mathematical expression for the sum of all terms between the 10th term and the 15th term inclusive.

Calculator Free

6. [4 marks]

The general rule of a geometric sequence is given by $T_n = \frac{4}{10^n}$, where $n = 1, 2, 3,$
Find the sum to infinity of this sequence if it exists. Justify your answer.

---

7. [4 marks: 2, 2]

(a) The sum of the first n terms of a geometric progression is given by $S_n = 5\times2.5^n - 5$. Determine the sum to infinity of this sequence if it exists.

(b) The sum of the first n terms of a geometric progression is given by $S_n = 0.25(1 - 0.2^n)$. Determine the sum to infinity of this sequence if it exists.

---

8. [3 marks]

The sum to infinity of a geometric sequence with first term 10 is 40. Find the recursive rule of this sequence.

Calculator Assumed

9. [10 marks: 3, 2, 2, 3]

A sequence is described by the rule $T_n = 1500(1.04)^n$, where $n = 1, 2, 3, \ldots$.

(a) State the recursive rule of this sequence.

(b) Find the first term that exceeds 2 000.

(c) Find the least value for n for which the sum of the first n terms is greater than 50 000.

(d) Find the sum of the second set of ten terms.

Calculator Assumed

10. [6 marks: 3, 3]

The fourth term and the ninth term of a geometric sequence are respectively 2662 and 428 717 762.

(a) Find the common ratio of this sequence.

(b) Find the sum of the first six terms of this sequence.

---

11. [5 marks: 2, 3]

In its first year of operation, a recycling plant processed 2 000 tonnes of recyclable waste **per month**. In the second year it processed 3000 tonnes per month and in the third year it processed 4500 tonnes per month. Successive amounts form a geometric sequence until year five when it reached its processing capacity.

(a) What was the maximum processing capacity of the recycling plant per month?

(b) Determine the total amount of waste processed for the first **five years**.

Calculator Assumed

12. [10 marks 2, 2, 2, 2, 2]

Peter hopes to save enough money to buy himself a new computer. In his plan, which covers a period of a fortnight, he saves 10 cents on the first day, 20 cents on the second day, 40 cents on the third day etc. , each time doubling the amount he saved the previous day.

(a) How much would Peter have to save on the 10th day?

(b) How much would Peter have saved by the end of the 10th day?

(c) On which day would he have to save $3.20?

(d) How many days would he take to save a total of at least $20.00?

(e) Comment on whether his savings plan is realistic.

13. [7 marks: 2, 1, 2, 2]

A toad hops 200 m along an outback highway on the first night. The distance hopped each night is 1% less than the distance hopped the previous night.

(a) How far does the toad hop during the 6th night?

Calculator Assumed

13. (b) How far would the toad have hopped by the end of the 6th night?

(c) Find the least number of nights required for the toad to have hopped a total distance of at least 10 km. Justify your answer.

(d) Will the toad be able to reach a roadhouse 25 km from where it started hopping? Justify your answer.

14. [10 marks: 2, 3, 3, 2]

An observation balloon is released from a height of 100 metres and allowed to float vertically upwards. The height increase in the first minute is 10 metres. Thereafter, the height increase during each subsequent minute is 70% of the height increase during the previous minute. Ignore air and wind resistance.

(a) Find the height increase during the 8th minute.

(b) Find the height at the end of the 8th minute.

(c) Find when the height of the balloon first exceeds 120 metres.

(d) Determine with reasons if the balloon will ever reach a height of 135 m.

Calculator Assumed

15. [13 marks: 2, 3, 2, 3, 3]

A robot submarine was released from a depth of 100 metres. During the first minute it descends a vertical distance of 20 metres. During the second minute it descends (vertically) a further 16 metres. The vertical descent each subsequent minute is 80% of the descent of the previous minute.

(a) Find the vertical descent during the 5th minute.

(b) Find the depth of the robot submarine at the end of the 5th minute.

(c) During which minute did the robot submarine descend by 2.147 m?

(d) Find when the robot submarine reached a depth of 180 metres.

(e) The seabed is 250 metres below sea level. Will the robot submarine ever reach the sea bed? Justify your answer.

Calculator Assumed

16. [10 marks: 2, 2, 3, 3]

A rubber ball is dropped from a height of 200 cm. Each time it hits the ground it will bounce vertically upwards to a height that is 80% of the height it reached in the previous bounce. It bounced to a height of 150 cm after it hit the ground the first time.

(a) Find the height reached by the ball after it hits the ground for the 3rd time.

(b) After how many times would the ball have to hit the ground before it first rebounds to a height less than 50 cm.

(c) Find the total distance travelled by the ball just before it hits the ground for the 5th time.

(d) Find the total distance travelled by the ball before it comes to rest on the ground.

---

17. [7 marks: 2, 3, 2]

An investment account pays 15% interest compounded annually over a 15 year period. Brad invests $100 000 in this account for 15 years. No new money was added to and no withdrawals were made from the investment account.

(a) Calculate the value of the investment account after 10 years.

Calculator Assumed

17. (b) Calculate the increase in value of the account during the 10th year.

(c) Calculate the minimum number of years required for the initial amount invested to double.

---

18. [10 marks: 2, 3, 3, 2]

$1 000 000 is invested in an account that pays interest at a rate of 5% per annum compounded annually. Let B(n) be the account balance at the end of n years.

(a) Find the general rule for the account balance at the end of n years.

(b) Find the growth in the account balance in the first 10 years.
Hence, find the average percentage growth rate in the first 10 years.

(c) Calculate the average percentage growth rate in the first 20 years.

(d) Give an explanation for the different answers in parts (b) and (c).

Calculator Assumed

19. [9 marks: 3, 3, 3]

To fight an infection, Steele has to take a course of medication which consists of 10 tablets to be taken over 10 days. One tablet is to be taken at the same time each day. Each tablet contains 50 mg of a particular drug. At the end of each 24 hour period, only 20% of the drug remains in the body. The table below models the amount of the drug in the body for a period of 5 days.

Day	Amount of drug in the body (mg)	
	Just before tablet is taken	Just after the tablet is taken
1	0	50
2	50×0.2	$50 + 50 \times 0.2$
3	$(50 \times 0.2) + (50 \times 0.2^2)$	$50 + (50 \times 0.2) + (50 \times 0.2^2)$
4	$(50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3)$	$50 + (50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3)$
5	$(50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3) + (50 \times 0.2^4)$	$50 + (50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3) + (50 \times 0.2^4)$

(a) Find the amount of drug left in the body just before the 6th tablet is taken.

(b) Find the amount of drug in the body just after the 10th tablet is taken.

(c) Find the amount of drug left in the body one week after the last tablet was taken. Comment on your answer.

Calculator Assumed

20. [14 marks: 1, 4, 4, 5]

A group of engineers plan to bore a horizontal tunnel underneath a city to accommodate a railway line. The tunnel needs to be 3 km long. Assume that the tunnel is in a straight horizontal line. A tunnel boring machine is assembled and used to bore the tunnel. On the first ten days, the machine bores a distance of 100 m each day. Between the 11th and 20th day inclusive, because of the fragile nature of the environment the distance bored each day is 90% of the distance bored in the previous day (the length bored is 90 m on the 11th day). From and on the 21st day onwards, the machine bores an extra 10 m more than what was bored in the previous day.

(a) Find the total distance bored in the first 10 days.

(b) How much *less distance* is bored on the 20th day compared to the 19th day?

(c) Find the total distance bored in the first 20 days (2 decimal places). Show full working out.

(d) How many days are required to complete boring the entire 3 km tunnel? Justify your answer.

26 Exponential Functions II

Calculator Assumed

1. [6 marks: 1, 2, 3]

   Q, the number of organisms (*in hundreds*) in a laboratory culture is related to time t (days) by the formula $Q = 16 \times (1.075)^t$.

   (a) How many organisms were there at the start?

   (b) Find the number of organisms after one week.

   (c) How long will it take for the population to reach 2 000?
   Show clearly the method you used. Give your answer to the nearest day.

---

2. [7 marks: 1, 3, 3]

   The amount of radioactive substance at time t years is given by $A = 150\ (0.8)^{t+1}$ g.

   (a) How much radioactive substance was there at the start?

   (b) Find the amount of radioactive substance that has decayed after 10 years.

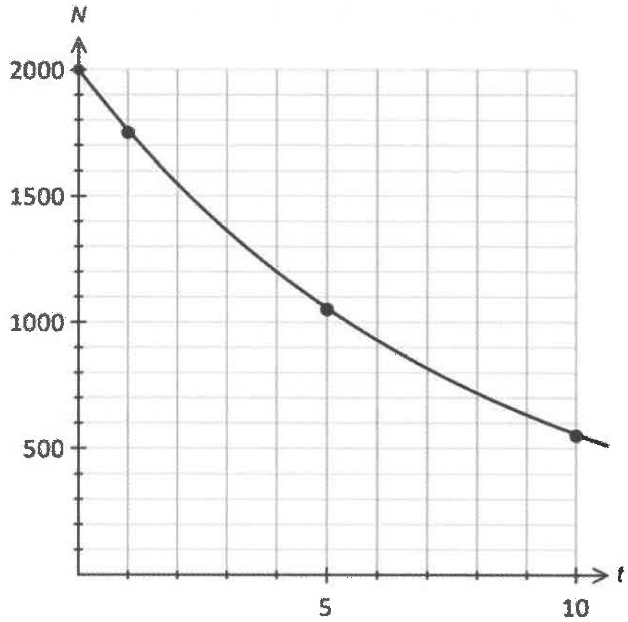
   (c) How long will it take for half the original amount to decay?
   Show clearly the method you used. Give your answer to the nearest year.

Calculator Assumed

3. [10 marks: 2, 1, 2, 5]

At the commencement of the use of a desalination plant, the number of dolphins at a cove near the desalination plant was estimated to be 2000. The number of dolphins, N, was monitored for several years and is displayed in the table below.

Years after, t	Number of dolphins, N
0	2000
1	1750
5	1050
10	550



The accompanying graph plots the points from the given table onto a set of axes.

The relationship between N and t is of the form $N = a(k^t)$.

(a) Use an appropriate method to find the values of a and k.
Give the value of a to the nearest 100 and the value of k to 2 decimal places.

(b) Predict the population after 20 years.

(c) How many years will it take for the dolphin population to reach 100?

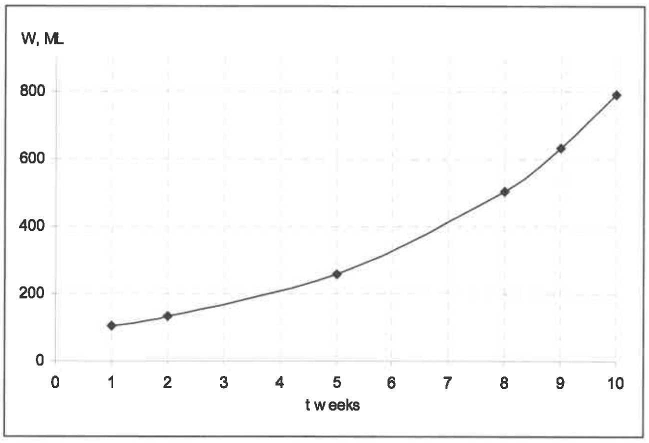
(d) The population of another marine animal in the area was also studied over the same time period and its population, Q, is given as $Q = 500\,(1.05)^{t+1}$.
Draw the graph of Q onto the diagram given and use the graphs drawn to determine when the two populations are equal?

Calculator Assumed

4. [9 marks: 3, 2, 3, 1]

The amount of water, W MegaLitres, in a newly constructed dam at time t weeks is shown in the graph below. Three models were suggested for this data:

$$W = 80 \times 1.25^{t}$$
$$W = 100 \times 1.26^{t}$$
$$W = 50 \times 1.26^{t}$$



(a) Which of these three models best represent the data given? Justify your answer.

(b) Use your chosen model to:

(i) estimate the amount of water in the dam after 20 weeks.

(ii) find when the amount of water in the dam will first exceed 3 000 ML.

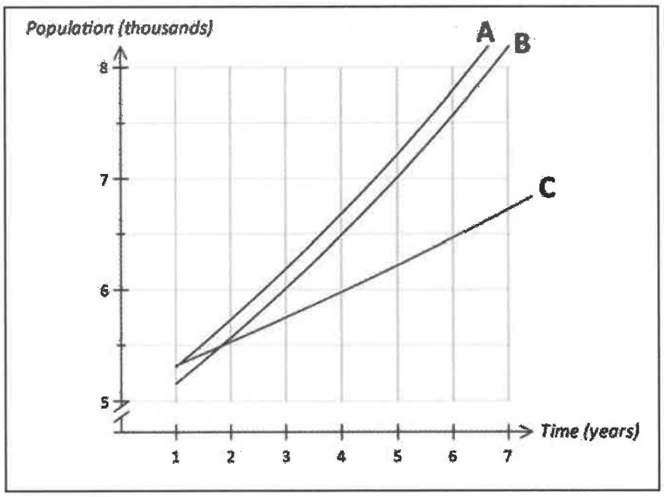
(c) What is the most important assumption underlying the model you chose in part (b)?

Calculator Assumed

5. [12 marks: 3, 2, 2, 2, 3]

The fox population (in thousands) in a forest in the South West of the state is displayed in the table below:

After t years	Population, P (thousands)
1	5.3
2	5.5
3	5.9
4	6.4
5	6.9
6	7.8



Three models are proposed to describe the fox population as tabulated.

$P = 4.91 \times 1.08^t$ $\quad P = 5.11 \times 1.04^t$ $\quad P = 4.77 \times 1.08^t$

The graphs of the proposed models are drawn above.

(a) Match the curves drawn with the models given.

A. ________________ B. ________________ C. ________________

(b) Plot the values of P as displayed in the table onto the given graph.

(c) Which of the three given models best describe the tabulated data. Explain clearly how you arrived at your answer.

(d) Use your chosen model to find:
(i) the population after 10 years.

(ii) when the population reaches 9 000 (*Give answer to the nearest month*).

27 Differentiation

Calculator Free

1. [10 marks: 1, 1, 2, 3, 3]

Differentiate with respect to x:

(a) $3x^2 - 4x + 10$

(b) $\dfrac{2}{3x^2} - x - 1$

(c) $\dfrac{1}{2\sqrt{x}} + \dfrac{5\sqrt{x}}{2}$

(d) $\dfrac{3x^4 + x^3}{4x}$

(e) $(2x + 1)^3$

2. [4 marks: 2, 2]

Given $f(x) = x^3 + x^2 - x + 4$, find:

(a) $f'(1)$

Calculator Free

2. (b) x if $f'(x) = 0$.

---

3. [2 marks]

Find the gradient of the curve $y = x^2 + 2\sqrt{x} + 1$ at the point where $x = 1$.

---

4. [5 marks]

Find the equation of the tangent to the curve $y = \dfrac{x^2 - x^3}{x^4}$ at the point where $x = -1$.

---

5. [5 marks]

Find the coordinates of the point(s) on the curve $y = \dfrac{1}{x} + x$ with a gradient of 0.

Calculator Free

6. [5 marks]

The curve $y = ax^3 + bx^2 + 4x + 1$ has a gradient of 2 at the point $(-1, -4)$.
Find a and b.

---

7. [6 marks]

Given that $y = ax^3 + bx^2 + 2$ has a tangent with equation $y = -4x + 5$ at the point where $x = 1$, find a, and b.

Calculator Free

8. [6 marks]

A curve has equation $y = \frac{x^3}{3} + \frac{x^2}{2} - 4x + 1$. The points A and B lie on this curve and the tangents to the curve at A and B are parallel to the line $2x - y = 5$. Find the coordinates of the points A and B.

9. [7 marks]

The tangent to the curve $y = x^3(x + 2)$ at the points where $x = 1$ and $x = -1$ meet at the point Q. Find the coordinates of the point Q.

Calculator Free

10. [9 marks: 6, 3]

Find the equation of the tangent(s) to the curve $y = \frac{x^3}{3} - x^2 - \frac{1}{3}$ that are:

(a) parallel to the line $x + y = 6$.

(b) perpendicular to the line $x - y = 1$.

---

11. [8 marks]

A curve has equation $y = (x - 2)(2x^2 - 5x + 2)$. The points A and B lie on this curve. The tangents to the curve at A and B are parallel to the line $12x - y = 5$. Find the coordinates of the points A and B.

Calculator Free

12. [3 marks]

Use first principles to determine the derivative of $y = 5x^2$.

---

13. [4 marks]

Use first principles to determine the derivative of $y = \dfrac{1}{x^2}$.

Calculator Free

14. [2 marks]

Use an appropriate derivative to evaluate $\lim_{h \to 0}\left[\dfrac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\right]$.

---

15. [4 marks]

Use an appropriate derivative to evaluate $\lim_{h \to 0}\left[\dfrac{\left(1+\sqrt{5+h}\right)^2-\left(1+\sqrt{5}\right)^2}{h}\right]$

giving your answer in exact form. Show clearly how you obtained your answer.

28 Derivatives & Graphs

Calculator Free

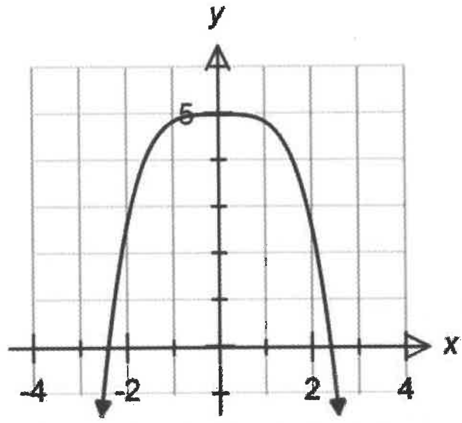
1. [6 marks: 2, 2, 2]

Use the sketch of $y = f(x)$ to determine the gradient of the curve at the points corresponding to the indicated values of x

.

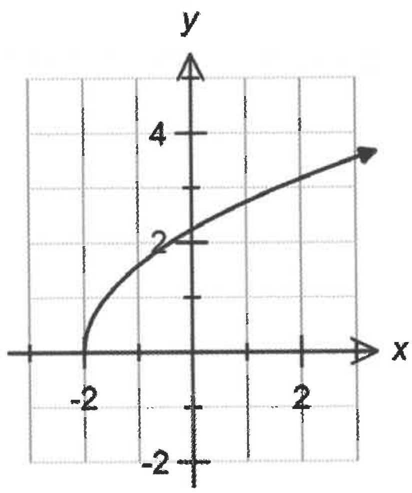
(a) (i) $x = -2$

(ii) $x = 1$



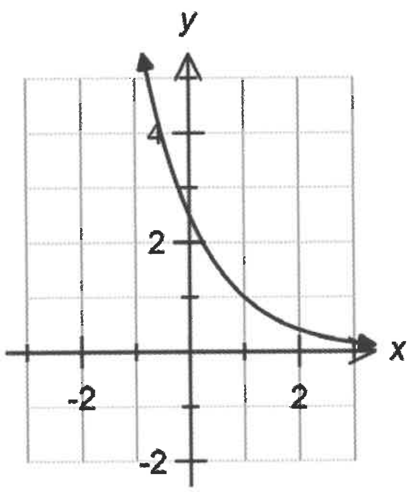
(b) (i) $x = -2$

(ii) $x = 0$



(c) (i) $x = 0$

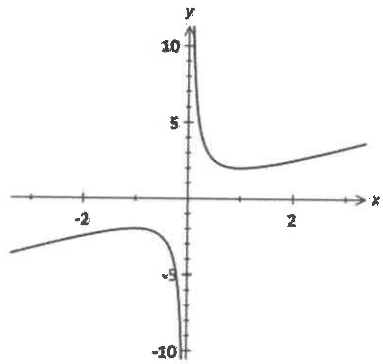
(ii) $x = 2$



Calculator Free

2. [4 marks: 2, 2]

The graph of $y = f(x)$ is given in the accompanying diagram.

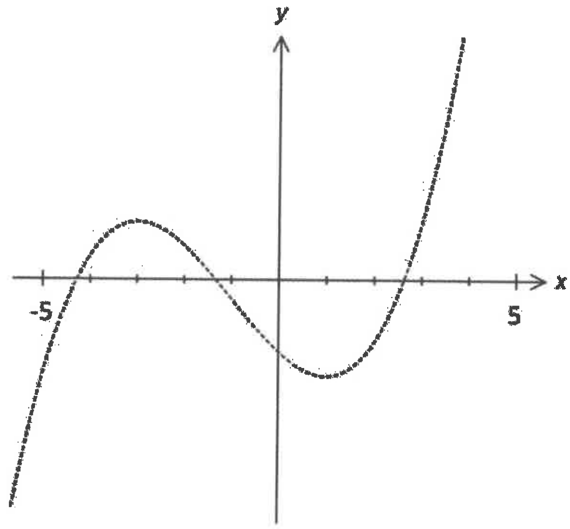


(a) Find the x-coordinate of the point(s) where the gradient of the curve is 0.

(b) For what values of x is the gradient of the curve negative?

3. [5 marks: 2, 3]

The graph of $y = f(x)$ is given below.



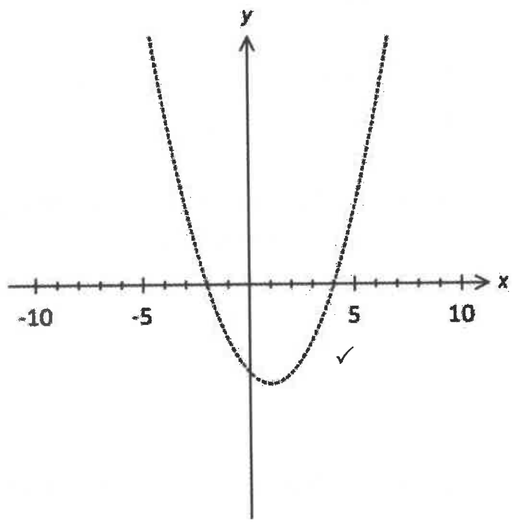
(a) For what values of x is the gradient negative?

(b) Sketch on the same axes, a possible graph of $y = f'(x)$.

Calculator Free

4. [6 marks: 2, 1, 3]

The graph of $y = f'(x)$ is given in the accompanying diagram.



(a) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is zero.

(b) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is a minimum.

(c) Sketch on the same axes a possible graph of $y = f(x)$.

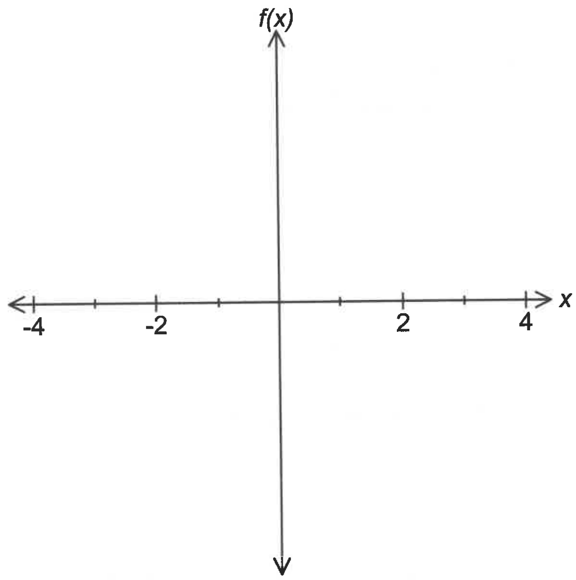
---

5. [4 marks]

The curve $y = f(x)$ cuts the x-axis at the origin and nowhere else.

$\frac{dy}{dx} = 0$ at $x = 1$ and $x = 2$. $\frac{dy}{dx} < 0$ only for $1 < x < 2$.

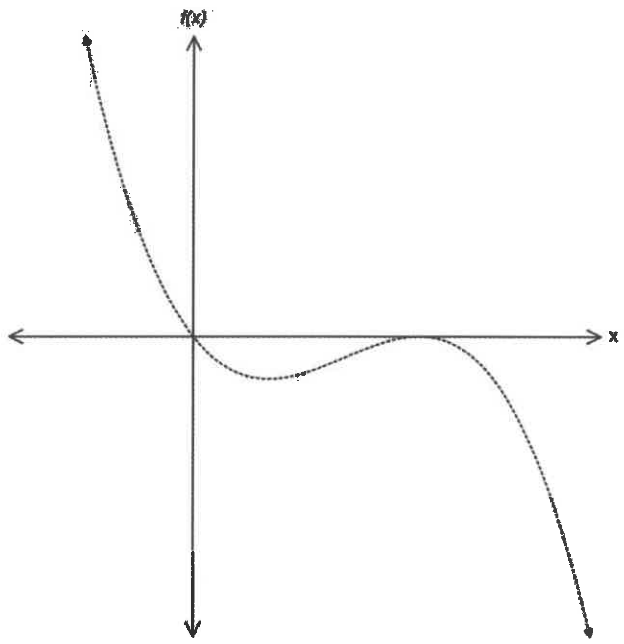
Give a possible sketch of $y = f(x)$.



Calculator Free

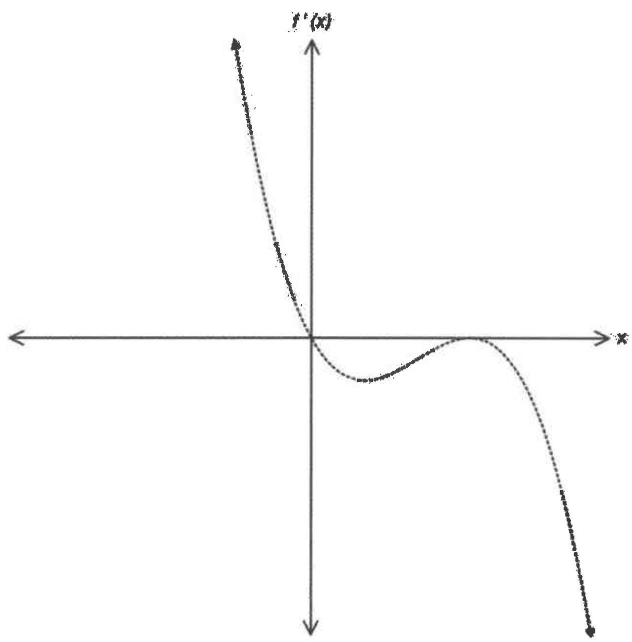
6. [3 marks]

Given the sketch of $y = f(x)$, on the set of axes given, give a possible sketch of $y = f'(x)$.



7. [3 marks]

Given the sketch of $y = f'(x)$, give a possible sketch of $y = f(x)$.



29 Stationary Points & Graphs

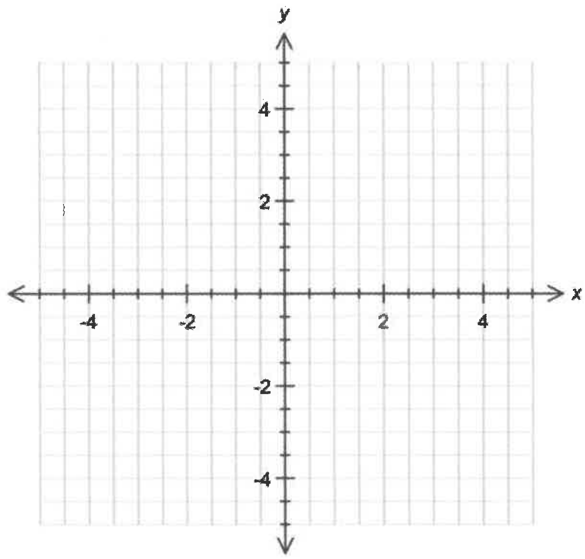
Calculator Free

1. [9 marks: 7, 3]

Consider the curve with equation $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{1}{3}$.

(a) Find the coordinates of the stationary point(s) on this curve. Use an appropriate analytical method to determine the nature of these point(s).

(b) Sketch the curve. Indicate clearly the turning points.



Calculator Free

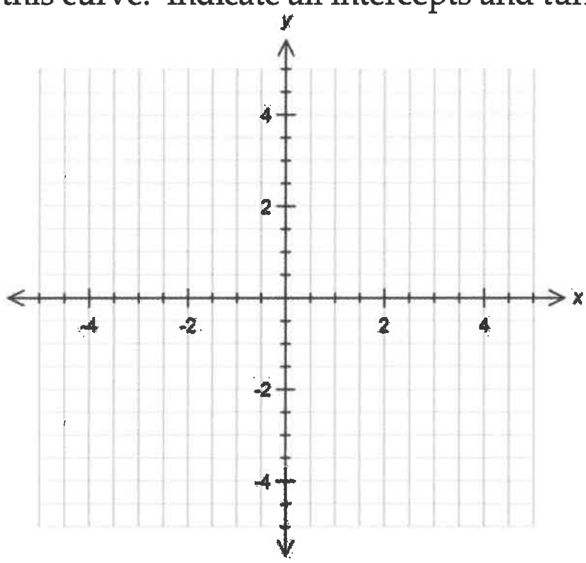
2. [14 marks: 3, 7, 4]

Consider the curve with equation $y = x^3 - 3x + 2$.

(a) Find the roots of this curve.

(b) Use a calculus method to determine the minimum and maximum points on this curve.

(c) Hence, sketch this curve. Indicate all intercepts and turning points.



Calculator Free

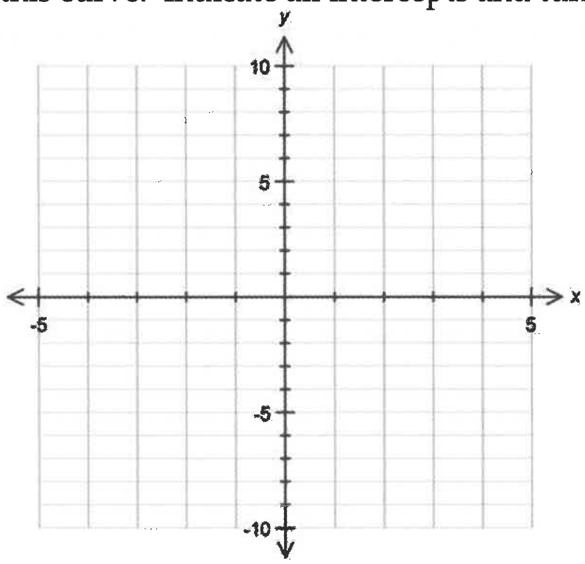
3. [11 marks: 5, 3, 3]

Consider the curve with equation $y = x^3 - 6x^2 + 12x - 9$.

(a) Find the coordinates of the stationary point(s) on this curve. Use a calculus method to determine the nature of these point(s).

(b) Find the coordinates of all the intercepts.

(c) Hence, sketch this curve. Indicate all intercepts and turning points.



Calculator Free

4. [5 marks]

The curve $y = 3x^3 + ax^2 + bx + c$ has a y-intercept at (0, 4) and stationary points at $x = 0$ and $x = 2$. Find the values of a, b and c.

5. [6 marks]

Consider the curve with equation $y = ax^3 + bx^2 - 12x + c$.
The curve has a turning point at (–1, 15) and another turning point at $x = 2$.
Find a, b and c. Show clearly how you obtained your answer.

Calculator Free

6. [10 marks]

A curve has equation $y = ax^3 + bx^2 + cx + d$. The curve has a turning point at $x = 1$, a y-intercept at $(0, -33)$ and a tangent with equation $y = -24x - 37$ at $x = -1$. Find the values of a, b, c and d. Show clearly how you obtained your answer.

Calculator Assumed

7. [8 marks]

A curve has equation $y = x^4 - x^2 - 4$.
Use a calculus method to find the coordinates of all the stationary points on this curve. Use the sign test to determine the nature of each of these points.

Calculator Assumed

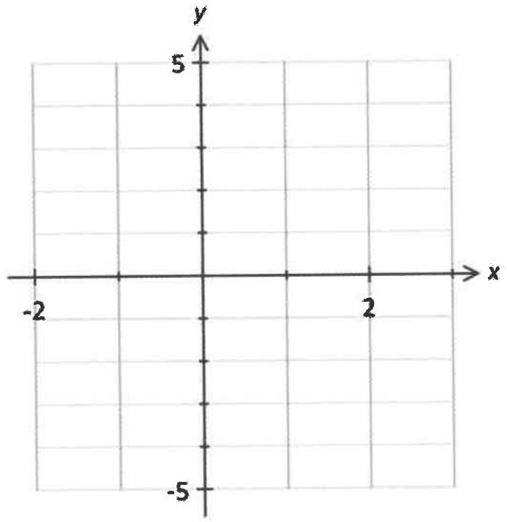
8. [10 marks: 2, 5, 3]

A curve has equation $y = (x - 2)(x^2 + 1)$.

(a) Find the coordinates of the horizontal and vertical intercepts.

(b) Use a calculus method to find the exact coordinates of the turning points. Use the sign test to determine the nature of these points.

(c) Sketch the graph of this curve in the axes below.



Calculator Assumed

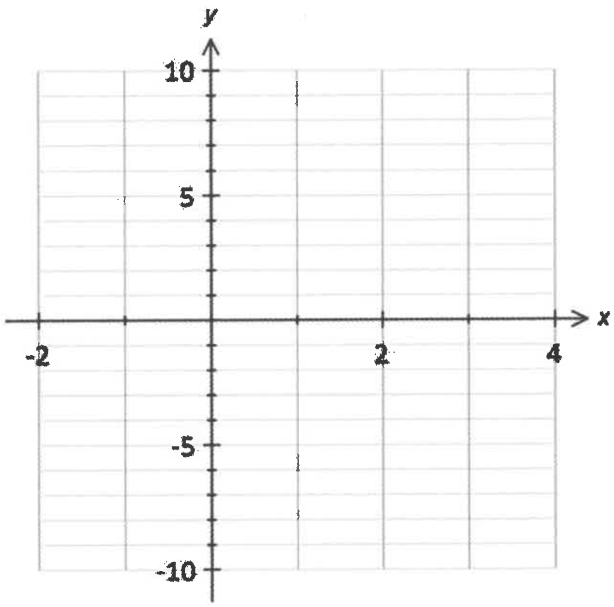
9. [10 marks: 2, 5, 3]

Consider the curve with equation $y = 2x^3 - 9x^2 + 12x - 4$.

(a) State the roots of this curve.

(b) Use a calculus method to determine the turning points on this curve. Use an appropriate method to determine the nature of each of these points.

(c) Sketch this curve for $0 \le x \le 3$ in the axes provided below. Label all intercepts and turning points.



30 Rates of Change

Calculator Assumed

1. [8 marks: 1, 2, 2, 3]

The height of a ball t seconds after it is thrown vertically upwards from ground level is given by $h = 20t - 5t^2$ metres.

(a) Find the height of the ball after 2 seconds.

(b) At what rate is the height of the ball changing when $t = 1$ second.

(c) Find when the rate of change of the height is zero.

(d) Find the height of the ball when the rate of change of height is $-10\ \text{ms}^{-1}$.

Calculator Assumed

2. [10 marks: 2, 1, 2, 1, 3, 1]

The volume (V cm^3) of a spherical balloon is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the balloon. The radius of the balloon changes with time t (seconds) according to the rule $r = 10 - t$.

(a) For what values of t is the rule $r = 10 - t$ valid? Why?

(b) Find V in terms of t.

(c) Find an expression for the rate with which the radius changes with time.

(d) Find the rate at which the volume is changing when $t = 5$ seconds. An exact answer is required.

(e) Find the exact value of t when the rate at which the volume changes is $-\pi$ cm^3s^{-1}.

(f) Hence, find the exact volume of the balloon when the rate at which the volume changes is $-\pi$ cm^3s^{-1}.

Calculator Assumed

3. [10 marks: 1, 2, 1, 1, 2, 3]

The mass (M g) of a crystal being grown in a laboratory at time t hours is given by $M = -\frac{1}{30}t^3 - \frac{1}{20}t^2 + 50t + 5$ for $0 \le t \le 20$.

(a) Find the change in mass of the crystal between $t = 0$ and $t = 10$ hours.

(b) Find the average rate of change of mass of the crystal in the first 10 hours.

(c) Find an expression for the instantaneous rate of change of mass of the crystal with respect to time.

(d) Find the instantaneous rate of change of mass of the crystal at $t = 10$ hours.

(e) Comment on the difference between your answers in part (b) and (d).

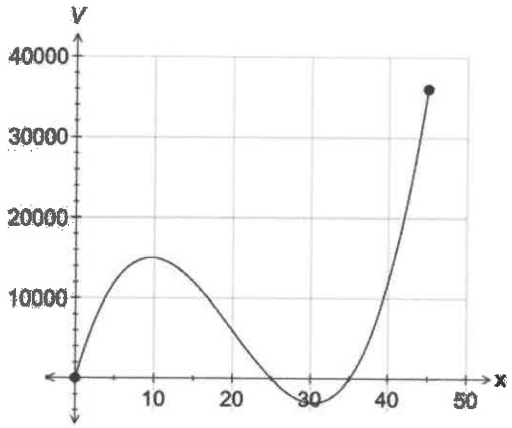
(f) Find the mass of the crystal when the instantaneous rate of change of mass is 48 g per hour.

31 Optimisation

Calculator Assumed

1. [4 marks: 2, 2]

The volume of a box of height x cm is given by $V = x(50 - 2x)(70 - 2x)$ cm^3. The graph of V against x is drawn below for $0 \le x \le 45$ cm.

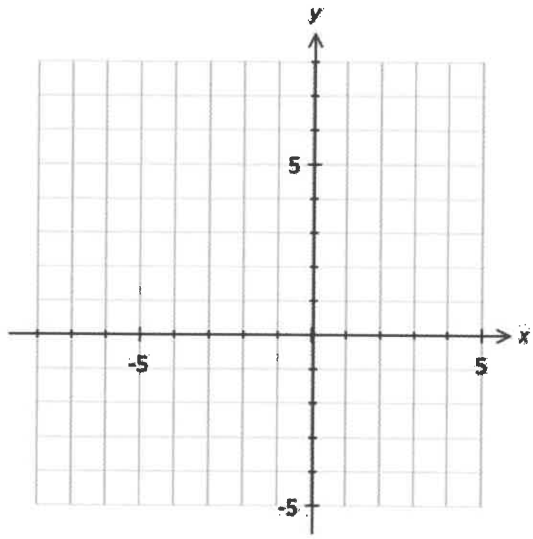


(a) Give the possible values of x.

(b) Use the graph to estimate the maximum possible volume of the box and give the associated value of x.

2. [6 marks: 4, 1, 1]

In the axes provided, sketch the graph of $f(x) = 0.1x^3 + 0.5x^2 - 1.2x - 3.6$ for $-7 \le x \le 4$. Indicate clearly the main features of the curve.



(a) Find to 2 decimal places the maximum value for $f(x)$ for $-7 \le x \le 0$.

(b) Find to 2 decimal places the minimum value for $f(x)$ for $-7 \le x \le 4$.

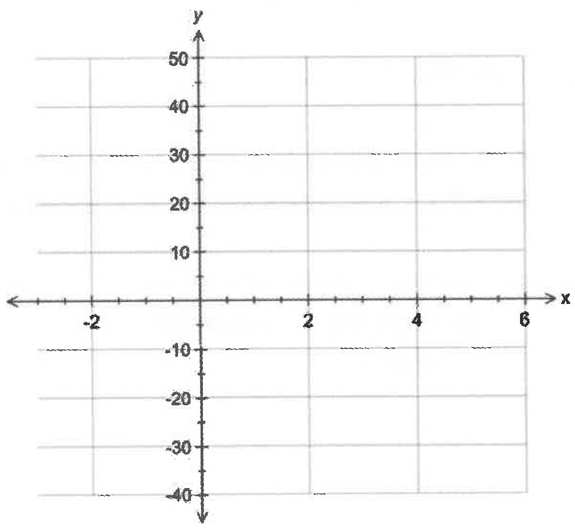
Calculator Assumed

3. [11 marks: 5, 3, 3]

Consider the curve with equation $y = (x-5)(x+1)^2 = x^3 - 3x^2 - 9x - 5$.

(a) Use a calculus method to determine the turning points on this curve.
Use an appropriate method to determine the nature of each of these points.

(b) Sketch this curve for $-3 \le x \le 6$ in the axes provided. Label all intercepts and turning points.



(c) For $y = (x-5)(x+1)^2 = x^3 - 3x^2 - 9x - 5$ in the domain $-3 \le x \le 6$, find:
(i) the largest value of y and the corresponding x-value

(ii) the lowest value for y and the corresponding x-value.

Calculator Assumed

4. [8 marks: 1, 5, 2]

The number of litres, V (kL), of unleaded petrol, sold at an outlet each day, is modelled by V = $\frac{t^3}{3} - 9t^2 + 65t + 50$, for $0 \le t \le 14$, where t is the number of days in the given fortnight. Use an analytical method to find:

(a) how many litres of unleaded petrol was sold at the start of the fortnight.

(b) the minimum amount of unleaded petrol sold per day and the corresponding t value.

(c) the maximum amount of unleaded petrol sold per day and the corresponding t value.

---

5. [8 marks: 1, 1, 6]

The organisers of a charity ball believe that if the ball tickets are priced at \$80 each they would be able to sell 500 tickets. For each \$5 increase in the price of each ticket, they expect the sales to decrease by 10 tickets.

(a) Find the number of tickets they expect to sell if the price of each ticket is increased by x lots of \$5.

(b) Find the expected revenue when the price of each ticket is raised by x lots of \$5.

Calculator Assumed

5. (c) Use Calculus to find the price per ticket that will maximize the revenue of the organisers. Give the maximum revenue.

---

6. [8 marks: 6, 2]

The population of dingoes in a large nature reserve is modelled by
$P = t^3 - 35t^2 + 275t + 875$ for $0 \le t \le 25$, where t is time in years after Jan 2000.

(a) Use Calculus to find the population at its lowest level. Give the year when this occurred.

(b) Use Calculus to find the population at its highest level between 2000 and 2025 inclusive. Give the year when this occurred.

Calculator Assumed

7. [8 marks: 3, 6]

A rectangular sheet of cardboard, 10 cm by 15 cm, is to be made into an open rectangular box. Four squares, each of side, x cm, are removed from each corner of the cardboard to form the net of the box.

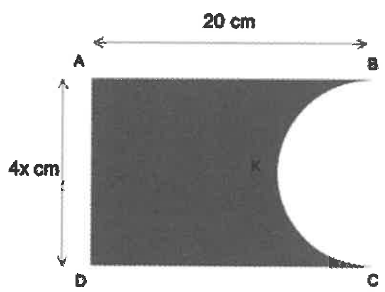
(a) Show that the volume, V, of the box is given by $V = x(15 - 2x)(10 - 2x)$.

(b) Use a calculus method to find the dimensions of the box that will maximise its volume.

---

8. [6 marks: 2, 4]

The figure shown in the diagram is obtained by removing a semi-circle BKC from the rectangle ABCD. AB = 20 cm and AD = $4x$ cm. The perimeter of the figure ABKCD is 200 cm.



(a) Show that the area of figure ABKCD is given by $A = 80x - 2\pi x^2$ cm^2.

Calculator Assumed

8. (b) Use calculus techniques to find in terms of π, the value of x that will maximise A. State this maximum value, in terms of π.

---

9. [8 marks]

A closed rectangular box, has a volume of 10 000 cm^3. The height of the box is twice its width. Use a calculus method to find the dimensions of the box that will minimise its surface area.

Calculator Assumed

10. [10 marks: 4, 6]

The total surface area of a closed rectangular box is 2 000 cm^2. The length of the box is four times its height x cm.

(a) Show that the volume of the box is given by $V = 800x - 3.2x^3$

(b) Use a calculus method to find the maximum volume of the box and the corresponding dimensions of the box.

32 Anti-Differentiation

Calculator Free

1. [9 marks: 1, 1, 1, 2, 2, 2]

Find the anti-derivative of each of the following:

(a) $3x^2 + 4$

(b) $\dfrac{x^3}{2}$

(c) $\dfrac{4x^3}{5}$

(d) $\dfrac{3x^2 + 5x^3}{4}$

(e) $(x^2 + 1)^2$

(f) $\dfrac{-2x^5 + 5x^4}{3x^2}$

Calculator Free

2. [3 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = 2x + 5$. Find the equation of the curve if it passes through $(-1, 3)$.

---

3. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = -\frac{4x^3}{3} + 2x - 1$. Find the equation of the curve if it passes through $(1, 2)$.

---

4. [4 marks]

Find $f(x)$ if $f'(x) = x^2 + 2x + k$ and $f(0) = -2$ and $f(-1) = \frac{-1}{3}$.

33 Rectilinear Motion

Calculator Assumed

1. [11 marks: 1, 2, 2, 3, 3]

A particle P moves along a straight line. Its displacement s metres, t seconds after passing a fixed point O is given by $s = -3t^2 + 4t + k$.

(a) Find the value of k .

(b) Find an expression for the velocity t seconds after passing O.
Hence, find the velocity of P as it passed O the first time.

(c) Find when P is travelling with a velocity of 2 ms^{-1}.

(d) Find when P is travelling with a speed of 2 ms^{-1}.

(e) Find when P is 1 metre away from O.

Calculator Assumed

2. [10 marks: 1, 2, 2, 1, 2, 1, 1]

The displacement of a particle moving along a straight line at time t seconds is given by $s = t^3 - \frac{9}{2}t^2 + 6t$ metres.

(a) Find the displacement of the particle at time $t = 1$ seconds.

(b) Find the change in displacement in the first 2 seconds.

(c) Find the velocity of the particle at $t = 2$ seconds.

(d) Find when the particle changes direction.

(e) Find the distance travelled in the first two seconds.

(f) Find the average speed in the first two seconds.

(g) What does the difference between your answers in (b) and (e) imply?

Calculator Assumed

3. [7 marks: 1, 3, 1, 2]

The displacement of a body at time t seconds is given by $s = 4t + \dfrac{1}{1+t}$ metres.

(a) Find an expression for the velocity of the body at time t seconds.

(b) Show that the body is never stationary.

(c) Find an expression for the acceleration at time t seconds.

(d) Hence, describe the motion of the body for large values of t.

Calculator Assumed

4. [10 marks: 4, 2, 4]

The displacement of a body moving along a straight line is given by $s = -t^3 + at^2 + bt + 3$ metres where t is time in seconds. The initial velocity of the body is 5 ms^{-1}. The body is momentarily at rest when $t = 1$ second.

(a) Find the values of a and b.

(b) Find when the body changes direction.

(c) Find the instantaneous speed at $t = 2$ seconds and the average speed in the first 2 seconds.

Calculator Assumed

5. [6 marks: 2, 2, 2] [TISC]

The velocity v cms^{-1}, of a particle P moving in a straight line at a point x cm from the origin is given by the equation $v^2 = -\int x \, dx$. P starts from the origin with a velocity of 10 cms^{-1}.

(a) Show that $v^2 = 100 - \dfrac{x^2}{2}$.

(b) Find where P is instantaneously at rest.

(c) Find the maximum speed of P and state where it occurs.

Calculator Assumed

6. [5 marks: 2, 3]

The acceleration (ms^{-2}) of a particle moving along a straight line is given by $a = 4t + 1$, where t is time in seconds. At $t = 1$, the velocity of the particle is $5\ ms^{-1}$ and the displacement of the particle is 10 m.

(a) Find an expression for the velocity of the particle at any time t.

(b) Find an expression for the displacement of the particle at any time t.

Calculator Assumed

7. [10 marks: 1, 6, 3]

A particle starts off from a fixed point 0 with an acceleration (mms^{-2}) of $a = mt - 24$, where t is time in seconds. The particle travels in a straight line and returns to O at $t = 4$ seconds and has a change of displacement of -9 mm in the third second (it moves in the same direction during this time).

(a) Find in terms of m an expression for the velocity of the particle at any time t.

(b) Find the displacement of the particle at any time t.

(c) Find when the particle is at O the third time (if it does).

Fully Worked Solutions

01 Lines

Calculator Free

1. [5 marks: 1, 2, 2]

A line passes through the points (1, 2) and (5, 22).

(a) Find the gradient of this line.

$m = \frac{22-2}{5-1} = 5$ ✓

(b) Find the equation of this line.

$y = 5x + c$
Subst. $x = 1, y = 2 \Rightarrow 2 = 5 + c$
$c = -3$ ✓
Hence, $y = 5x - 3$ ✓

(c) Is (3, 25) on this line? Justify your answer.

Subst. $x = 3$ into equation of line.
$\Rightarrow y = 5 \times 3 - 3 = 12$
Hence, when $x = 3, y = 12 \neq 25$. ✓
Therefore, (3, 25) is not on this line. ✓

2. [3 marks]

The points (–2, 5), (3, *k*) and (5, 12) are collinear. Find the value(s) of *k*.

Gradient $= \frac{k-5}{3-(-2)} = \frac{12-5}{5-(-2)}$ ✓✓
$\frac{k-5}{5} = \frac{7}{7}$
$k = 10$ ✓

3. [3 marks]

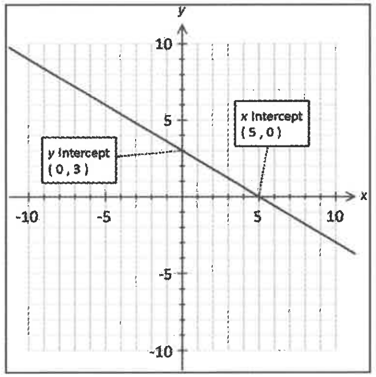
The line $ax + by = 18$ passes through the point (1, –4) and has a gradient of 2. Find a and b.

Gradient $= \frac{-a}{b} = 2 \Rightarrow a = -2b$ ✓
Subst. $x = 1, y = -4$ into equation of line:
$-2b(1) + b(-4) = 18$
$-6b = 18 \Rightarrow b = -3$ ✓
Hence, $a = 6$ ✓

Calculator Free

4. [2 marks]

Sketch the line with equation $3x + 5y = 15$.
Indicate clearly on your sketch the intercepts of this line.



Correct x-intercept: ✓
Correct y-intercept: ✓

---

5. [6 marks: 3, 3]

Find the equation of the line passing through the point with coordinates (10, 3):

(a) and parallel to the line with equation $4x + 5y = 20$.

$4x + 5y = 20 \Rightarrow$ gradient $= -4/5$ ✓
Hence: $y = -4x/5 + c$
Subst. $x = 10, y = 3 \Rightarrow 3 = -8 + c$ ✓
$c = 11$
Hence, $y = -4x/5 + 11$ ✓

(b) and perpendicular to the line with equation $2x + 3y = 12$.

$2x + 3y = 12 \Rightarrow$ gradient $= -2/3$ ✓
Hence: $y = 3x/2 + c$ ✓
Subst. $x = 10, y = 3 \Rightarrow 3 = 15 + c$
$c = -12$
Hence, $y = 3x/2 - 12$ ✓

Calculator Free

6. [7 marks: 3, 2, 2]

The lines $2x + 3y = 12$ and $4x + 5y = 20$ meet at the point P.

(a) Find the coordinates of P.

$2x + 3y = 12$ (1)
$4x + 5y = 20$ (2)
(1) × 2 $4x + 6y = 24$ (3) ✓
(3) − (2) $y = 4 \Rightarrow x = 0$ ✓
Hence, P(0, 4). ✓

(b) Find the equation of the line through P and parallel to the line with equation $2x + y = 10$.

Gradient of given line = −2 ✓
Gradient of required line = −2
Equation of required line: $y = -2x + 4$ ✓

(c) Find the equation of the line through P and perpendicular to the line with equation $2x + y = 10$.

Gradient of given line = −2
Gradient of required line = $\frac{1}{2}$ ✓
Equation of required line: $y = \frac{1}{2}x + 4$ ✓

---

7. [3 marks: 2, 1]

Consider the line $2x + by = c$ where c is a constant.

(a) Find b if this line has gradient −4.

Gradient $= \frac{-2}{b} = -4$ ✓
$\Rightarrow b = \frac{1}{2}$ ✓

(b) Find c if this line has an x-intercept of 6.

When $y = 0, x = 6$
$\Rightarrow c = 12$ ✓

Calculator Free

8. [6 marks: 2, 2, 2]

Suggest one possible equation each for the lines L1 and L2 if:

(a) L1 and L2 are each parallel to $x + 2y = 0$.

> Example:
> L1: $x + 2y = 4$ L2: $x + 2y = 10$ ✓✓

(b) L1 and L2 meet at the point with coordinates (0, 4) and are perpendicular to each other.

> Example:
> L1: $y = 2x + 4$ L2: $y = -x/2 + 4$ ✓✓

(c) L1 and L2 do not intersect.

> Any two parallel lines.
> For example $x + y = 1$ and $x + y = 2$. ✓✓

9. [5 marks: 3, 2]

The line with equation $7x + 5y = 70$ intersects the x-axis and y-axis at A and B respectively.

(a) Find the coordinates of the mid-point of AB.

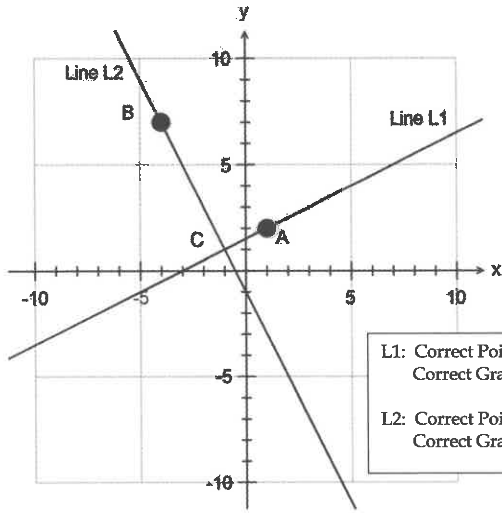
> At A: $y = 0 \Rightarrow x = 10$.
> Hence A(10, 0). ✓
> At B: $x = 0 \Rightarrow y = 14$.
> Hence, B(0, 14). ✓
>
> Therefore, mid-point of AB has coordinates (5, 7). ✓

(b) Find the distance between A and B.

> A(10, 0) & B(0, 14).
> Therefore, AB = $\sqrt{10^2 + 14^2}$ ✓
> = $\sqrt{296}$. ✓

Calculator Free

10. [9 marks: 4, 5]



> L1: Correct Point ✓
> Correct Gradient ✓
>
> L2: Correct Point ✓
> Correct Gradient ✓

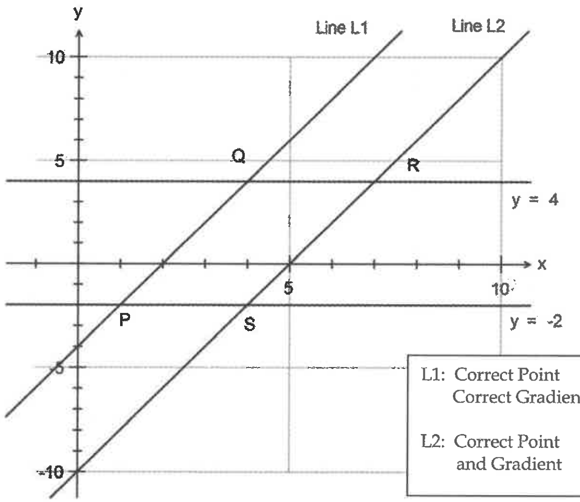
(a) On the axes provided above, sketch the line L1 which passes through the point A (1, 2) with gradient $\frac{1}{2}$. Also sketch the line L2 which passes through the point B (–4, 7) and perpendicular L1.

(b) Use your graph to find the coordinates of C the point of intersection between the lines in (a) and (b). Hence, find the area of Δ ABC.
[Hint: $\sqrt{5} \times \sqrt{45} = 15$]

> From graphs drawn, C(–1, 1). ✓
>
> AC = $\sqrt{1^2 + 2^2} = \sqrt{5}$ ✓
> BC = $\sqrt{6^2 + 3^2} = \sqrt{45}$ ✓
>
> Hence, area of ΔABC = $\frac{1}{2} \times \sqrt{5} \times \sqrt{45}$ ✓
> = $\frac{15}{2}$ square units. ✓

Calculator Free

11. [7 marks: 3, 2, 2]



L1: Correct Point	✓
Correct Gradient	✓
L2: Correct Point and Gradient	✓

The diagram above shows the lines with equations $y = -2$ and $y = 4$.

(a) On the diagram above draw the line L1 passing through the point (0, −4) with gradient 2. Also, draw the line L2 parallel to L1 but passing through the point (5, 0).

(b) The line L1 meets the lines $y = -2$ and $y = 4$ at P and Q respectively. The line L2 meets the lines $y = -2$ and $y = 4$ at S and R respectively.

(i) On the diagram above, clearly mark the points P, Q R and S. Find the area of PQRS.

> PQRS is a parallelogram.
> PS = 3 and Height of parallelogram = 6 ✓
> ⇒ Area of PQRS = 3 × 6 = 18 square units. ✓

(ii) Find the perimeter of PQRS.

> $PQ = \sqrt{3^2 + 6^2} = \sqrt{45}$. ✓
> Hence, perimeter of PQRS = $2(3 + \sqrt{45})$. ✓

Calculator Assumed

12. [10 marks: 2, 1, 3, 4]

Bill, a plumber charges a call-out fee of $100 plus $80 per half hour or part thereof. Ian, another plumber does not charge a call-out fee but charges $180 per hour or part thereof.

(a) How much will Bill charge for a job that is estimated to take exactly 4 hours?

> B = 100 + 80(2 × 4) ✓
> B = $740 ✓

(b) How much will Ian charge for a job that is estimated to take exactly 4 hours?

> I = 180 × 4
> = $720 ✓

(c) Determine, which plumber will be cheaper to employ if a job is estimated to take 3 hours and 20 minutes. Justify your answer.

> Bill will charge 7 lots of half-hour for the job.
> Hence, B = 100 + 80(7) = $660 ✓
> Ian will charge 4 lots of one-hour for the job.
> Hence, I = 180 × 4 = $720 ✓
> Hence, Bill will be cheaper to employ. ✓

(d) Under what conditions will it be cheaper to employ Bill? Justify your answer.

Time for Job, t hours	Bill's Charge	Ian's Charge
$0 < t \le 0.5$	100 + 80 = $180	$180
$0.5 < t \le 1$	100 + 2(80) = $260	$180
$1 < t \le 1.5$	100 + 3(80) = $340	180(2) = $360
$1.5 < t \le 2$	100 + 4(80) = $420	180(2) = $360
$2 < t \le 2.5$	100 + 5(80) = $500	180(3) = $540
$2.5 < t \le 3$	100 + 6(80) = $580	180(3) = $540
$3 < t \le 3.5$	100 + 7(80) = $660	180(4) = $720
$3.5 < t \le 4$	100 + 8(80) = $740	180(4) = $720
$4 < t \le 4.5$	100 + 9(80) = $820	180(5) = $900
$4.5 < t \le 5$	100 + 10(80) = $900	180(5) = $900
$5 < t \le 5.5$	100 + 11(80) = $980	180(6) = $1080
$5.5 < t \le 6$	100 + 12(80) = $1060	180(6) = $1080
$6 < t \le 6.5$	100 + 13(80) = $1140	180(7) = $1260
$6.5 < t \le 7$	100 + 14(80) = $1220	180(7) = $1260

From table, it is cheaper to employ Bill ✓✓
- for jobs that will take longer than 5 hours ✓
- for jobs that take longer than the first hour but do not exceed the first half hour of every hour. ✓

02 Quadratics

Calculator Free

1. [8 marks: 3, 2, 3]

A parabola has equation $y = (x-2)(5-x)$.

(a) Find the coordinates of the x and y intercepts of the parabola .

> x-intercepts (2, 0) and (5, 0) ✓✓
> y-intercepts (0, –10) . ✓

(b) Find the equation of the line of symmetry.

> Line of symmetry $x = \frac{2+5}{2} = 3.5$ ✓✓

(c) Find the coordinates of the turning point of the parabola and state the nature of the turning point.

> $y = (3.5-2)(5-3.5) = 2.25$ ✓
>
> Hence, turning point has coordinates (3.5, 2.25). ✓
> Turning point is a maximum point as x^2 coefficient is negative. ✓

---

2. [6 marks: 3, 3]

A parabola has equation $y = 10 - 6x - 3x^2$.

(a) Find the coordinates of the turning point and state its nature.

> Line of symmetry $x = \frac{-(-6)}{2(-3)} = -1$ ✓
> When $x = -1$, $y = 10 - 6(-1) - 3(-1)^2 = 13$ ✓
>
> Hence, turning point has coordinates (–1, 13).
> Turning point is a maximum point as x^2 coefficient is negative. ✓

(b) Find the exact x-intercepts.

> $3x^2 + 6x - 10 = 0$
> $x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)}$ ✓
> $= -1 \pm \frac{\sqrt{156}}{6} = -1 \pm \frac{\sqrt{39}}{3}$
>
> Hence, x-intercepts are $(-1 \pm \frac{\sqrt{39}}{3}, 0)$ ✓✓

Calculator Free

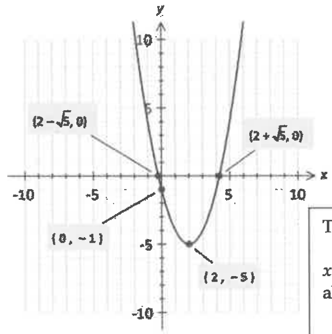
3. [6 marks: 3, 3]

A parabola has equation $y = (x-2)^2 - 5$.

(a) Find the exact coordinates of the x-intercepts.

> $(x-2)^2 = 5$
> $(x-2) = \pm\sqrt{5}$ ✓
> $x = 2 \pm \sqrt{5}$
> Hence, x-intercepts are $(2 \pm \sqrt{5}, 0)$ ✓✓

(b) Sketch this parabola. Indicate clearly the coordinates of the intercepts and the turning point.

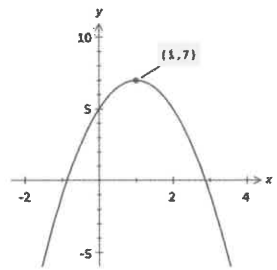


> Turning Point: ✓
>
> x-intercepts symmetrical about line of symmetry: ✓
>
> y-intercept: ✓

---

4. [3 marks]

Find the equation of the parabola shown in the accompanying diagram.

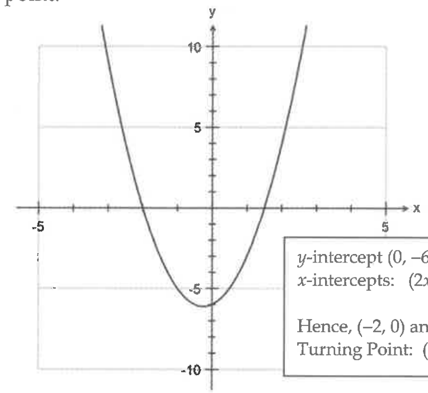


> $y = k(x-1)^2 + 7$ ✓
>
> When $x = 0$, $y = 5$;
> $5 = k + 7$
> $k = -2$ ✓
>
> Hence, equation is $y = -2(x-1)^2 + 7$ ✓

Calculator Free

5. [4 marks]

Sketch the parabola with equation $y = 2x^2 + x - 6$. Indicate clearly the intercepts and the turning point.



y-intercept $(0, -6)$. ✓
x-intercepts: $(2x-3)(x+2) = 0$
$x = -2, 3/2$
Hence, $(-2, 0)$ and $(3/2, 0)$. ✓✓
Turning Point: $(-1/4, -49/8)$. ✓

6. [4 marks]

A parabola has equation $y = -x^2 + 4x + 5$. Rewrite the equation of this parabola in the form $y = k(x-a)^2 + b$, giving the values of a, b and k.

$y = -(x^2 - 4x - 5)$ ✓
$= -[(x-2)^2 - 9]$ ✓
$y = -(x-2)^2 + 9$
Hence, $k = -1$, $a = 2$ and $b = 9$ ✓✓

7. [3 marks]

A parabola has equation $y = -x^2 + bx + c$. Find the values of b and c, if the parabola has a turning point at $(-1, 4)$ and an intercept at $(0, 3)$.

Line of symmetry $x = \frac{-b}{2(-1)} = \frac{b}{2}$ ✓
But line of symmetry is $x = -1$.
Hence, $\frac{b}{2} = -1 \Rightarrow b = -2$ ✓
y-intercept is at $(0, c)$. Hence, $c = 3$. ✓

Calculator Free

8. [4 marks]

A parabola has equation $y = k(x-a)(x-b)$ where k, a and b are constants with $a < b$. Find a, b and k if the parabola has an x-intercept at $(-3, 0)$, a turning point at $(1, 32)$ and a y-intercept at $(0, 30)$.

$a = -3$ ✓
Turning Point at $(1, 32)$
$\Rightarrow$ Line of symmetry has equation $x = 1$.
Hence, $\frac{-3+b}{2} = 1 \Rightarrow b = 5$ ✓✓

Equation of parabola is now $y = k(x+3)(x-5)$.
y-intercept $(0, 30) \Rightarrow 30 = k(3)(-5)$
$k = -2$ ✓

9. [6 marks: 1, 1, 2, 2]

Consider the parabola with equation $y = f(x) = (x-2)(x+a)$ where a is a constant.

(a) Find a if the parabola has exactly one root.

$a = -2$ ✓

(b) Find a if $f(2) = f(4) = 0$.

$a = -4$ ✓

(c) Find a if $f(0) = 10$.

$f(0) = -2a = 10$ ✓
$a = -5$ ✓

(d) Find a if the parabola has a turning point at $x = 3$.

Line of Symmetry $x = \frac{2-a}{2} = 3$ ✓
$a = -4$ ✓

Calculator Free

10. [12 marks: 2, 3, 3, 2, 2]

A parabola has equation $y = f(x)$ where $f(x) = k(x+a)^2 + 16$ where a is a constant.

(a) Find a and k if the parabola has a turning point at (1, 16).

$a = -1$ ✓
k = any real number ✓

(b) Find a and k if the parabola has a turning point at (–2, 16) and $f(0) = -4$.

$a = 2$ ✓
$f(0) = 4k + 16 = -4$ ✓
$k = -5$ ✓

(c) Find a and k if $f(3) = f(-5) = 0$.

LOS is $x = \frac{3+(-5)}{2} = -1$
$\Rightarrow a = 1$ ✓
$f(3) = k(3+1)^2 + 16 = 0$ ✓
$k = -1$ ✓

(d) Find k if the parabola has no roots.

For $k(x+a)^2 + 16 = 0$
$k(x+a)^2 = -16$
$(x+a)^2 = \frac{-16}{k}$ ✓
Since parabola has no roots, $k > 0$. ✓

(e) Explain clearly why the parabola cannot have exactly one root.

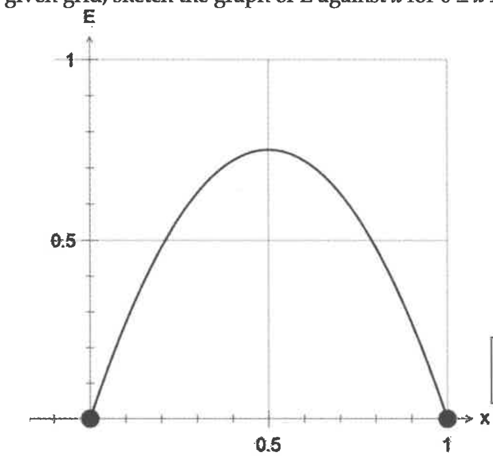
For $k(x+a)^2 + 16 = 0$
$k(x+a)^2 = -16$
$(x+a)^2 = \frac{-16}{k}$ ✓
For the parabola to have exactly one root,.
$\frac{-16}{k} = 0$. But this is impossible.
Hence, the parabola cannot have exactly one root. ✓

Calculator Assumed

11. [6 marks: 2, 1, 1, 2]

The efficiency rating, E, of a spark plug when the gap is set at x mm is given by $E = 3x(1-x)$.

(a) In the given grid, sketch the graph of E against x for $0 \le x \le 1$.



Roots ✓
Max Point ✓

(b) What values of x would give an efficiency rating of zero?

$x = 0$ and $x = 1$ (corresponding to the roots) ✓

(c) What is the value of the maximum efficiency rating?

Maximum E = 0.75 (From Calculator) ✓

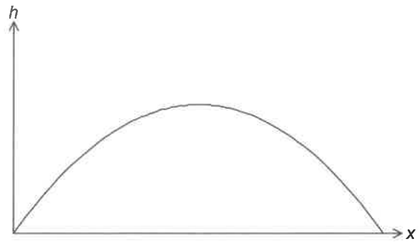
(d) Find the values of x between which the efficiency rating is 0.6 or more.

When E = 0.6, x = 0.28, 0.72
Hence, $E \ge 0.6 \Rightarrow 0.28 \le x \le 0.72$ ✓✓

Calculator Assumed

12. [6 marks: 1, 1, 2, 2]

The height (h metres) of a soccer ball in flight is given by $h = -0.008x(x - 40)$ for $x \geq 0$, where x (metres) is the horizontal distance travelled from the point where the ball was kicked. Assume that the ball travels in a vertical plane.



Use an appropriate method, stating clearly the method you have used, (either using algebra or using your CAS/graphic calculator) to find:

(a) the maximum height reached by the ball

Maximum $h = 3.2$ m (From Calculator) ✓

(b) the horizontal distance travelled by the ball if it was not intercepted during its flight

Roots $x = 0, 40$ (Algebraic)
Hence, horizontal distance travelled = 40 m. ✓

(c) the horizontal distance travelled by the ball if it was intercepted at a height of 2 metres.

When $h = 2$, $x = 7.75, 32.25$ (From Calculator) ✓
Hence, horizontal distance travelled
= 7.75 or 32.25 m. ✓

(d) x when the ball was more than 1 metre above the ground.

$-0.008x(x - 40) > 1$
From calculator: $3.42 < x < 36.58$. ✓✓

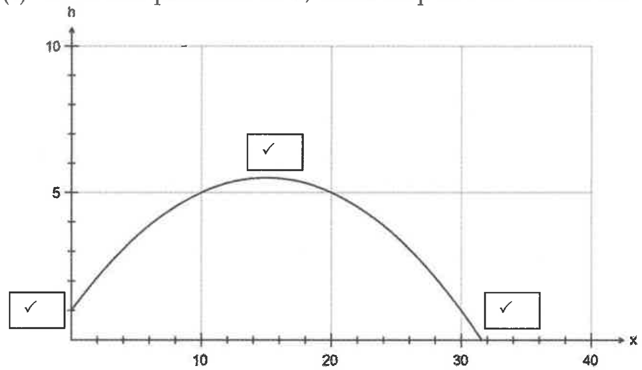
```
solve(-0.008x(x-40)>1,x)
        {3.4169<x<36.5831}
```


Calculator Assumed

13. [7 marks: 3, 1, 1, 2]

The height (h metres) of a cricket ball in flight is given by $h = 1 + \frac{3}{5}x - \frac{1}{50}x^2$ for $x \geq 0$, where x (metres) is the horizontal distance travelled from the point where the ball was struck by a bat. Assume that the ball travels in a vertical plane.

(a) On the axes provided below, sketch the path of the cricket ball.



Use an appropriate method, showing clearly the method you have used, (either using algebra or using your CAS/graphic calculator) to find:

(b) the height at which the ball was struck.

$x = 0$, $h = 1$.
Hence, the ball was struck when it was at a height of 1 metre. ✓

(c) the maximum height reached by the ball.

Maximum height is 5.5 m. ✓
[From calculator.]

(d) the horizontal distance travelled by the ball if it was caught when it was 2 m above the ground.

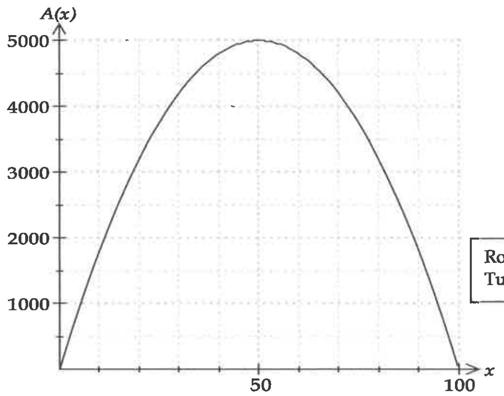
Use calculator to find intersection between
$h = 1 + \frac{3}{5}x - \frac{1}{50}x^2$ and $h = 2$.
$\Rightarrow$ $x = 1.77$ m or 28.23 m ✓✓

Calculator Assumed

14. [7 marks: 2, 1, 2, 2]

Gemma owns a hobby farm and needs to create a fenced up area for her sheep using the back wall of her shed as one of the sides of the fenced up area. She has 200 metres of fencing available. From what she could recall from her mathematics class when she was a student, to maximise the fenced up area, she would need to maximise the function $A(x) = x(200 - 2x)$ where x is the width of the fenced up area.

(a) On the axes provided below sketch $A(x) = x(200 - 2x)$.



Roots ✓
Turning Point ✓

(b) Find the coordinates of the turning point of function A(x).

Turning Point (50, 5000) ✓

(c) Find the maximum possible area that can be fenced and the dimensions of that fenced up area.

Maximum Fenced up Area = 5000 m^2. ✓
When width is 50 m and length is 100 m. ✓

(d) Find the possible dimensions of the fenced up area if its area is 3200 m^2.

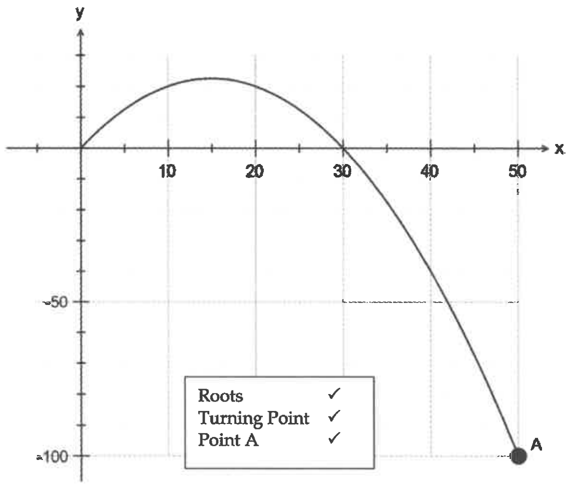
When $A = 3200$, $x = 20$ or 80.
Hence, possible dimensions: width = 20 m & length = 160 m ✓
or width = 80 m & length = 40 m ✓

Calculator Assumed

15. [8 marks: 3, 1, 1, 3]

A ball is thrown off the top of a cliff, 100 m above sea level. Taking the point of projection O as the origin of the coordinate axes, the path taken by the ball is given as $y = 0.1x(30 - x)$. The ball hits the surface of the sea at A.

(a) On the axes provided below, sketch the path of the ball. Mark the point A on your sketch.



Roots ✓
Turning Point ✓
Point A ✓

(b) Write the equation for the surface of the sea.

$y = -100$ ✓

(c) Find the distance from A to B, the base of the cliff.

A has coordinates (50, –100). ⇒ AB = 50 m ✓

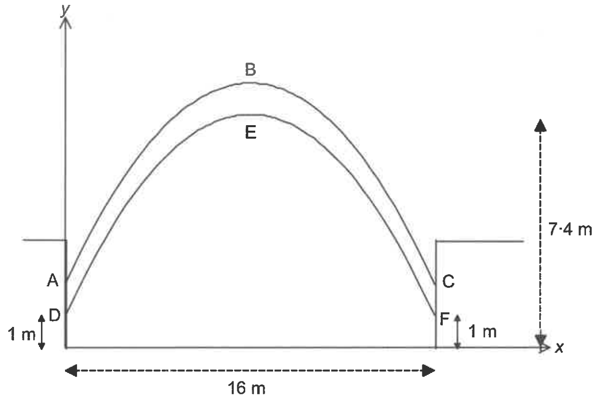
(d) Find the horizontal distance from O when the ball is 110m above sea level.

110 m above sea level ⇒ $y = 10$. ✓
From calculator, $y = 10$ ⇒ $x = 3.82, 26.18$
That is, 3.82 m and 26.18 m from O. ✓✓

Calculator Assumed

16. [9 marks: 3, 1, 1, 1, 3]

The diagram below shows the cross section of an arch. ABC and DEF are the top and bottom edges of the arch respectively. Each of these edges is approximately parabolic in shape. The edges ABC and DEF are "parallel" with ABC positioned 1 metre above DEF. D and F are each 1 metre above the road level. The road level is modelled by the x-axis. The vertical line through D and A is modelled by the y-axis.



(a) Find the equation of the bottom edge of the arch (DEF).

> Turning Point has coordinates (8, 7.4) ✓
> Hence, $y = k(x-8)^2 + 7.4$
> When $x = 0, y = 1 \Rightarrow 1 = k(-8)^2 + 7.4$
> $64k = -6.4$
> $k = -0.1$ ✓
>
> Hence, equation is $y = -0.1(x-8)^2 + 7.4$ ✓

(b) Find the equation of the top edge of the arch (ABC).

> Equation is $y = -0.1(x-8)^2 + 7.4 + 1$
> $y = -0.1(x-8)^2 + 8.4$ ✓

Calculator Assumed

16. (c) Find the coordinates of B, the highest point on the arch.

> Equation of top edge is $y = -0.1(x-8)^2 + 8.4$
>
> Hence, coordinates of B is (8, 8.4) . ✓

(d) What is the clearance of the arch (above road level) at a point which is horizontally 5 m from D?

> Equation of bottom edge is $y = -0.1(x-8)^2 + 7.4$
>
> When $x = 5$, $y = -0.1(5-8)^2 + 7.4$
> $= 6.5$ m ✓

(e) At what horizontal distance from D is the clearance of the arch 5 m above the road level?

> Equation of bottom edge is $y = -0.1(x-8)^2 + 7.4$ ✓
>
> When clearance = 5 m,
> $-0.1(x-8)^2 + 7.4 = 5$ ✓
> $-0.1(x-8)^2 = -2.4$
> $x = 3.10$ or 12.90 m. ✓

03 Cubics

Calculator Free

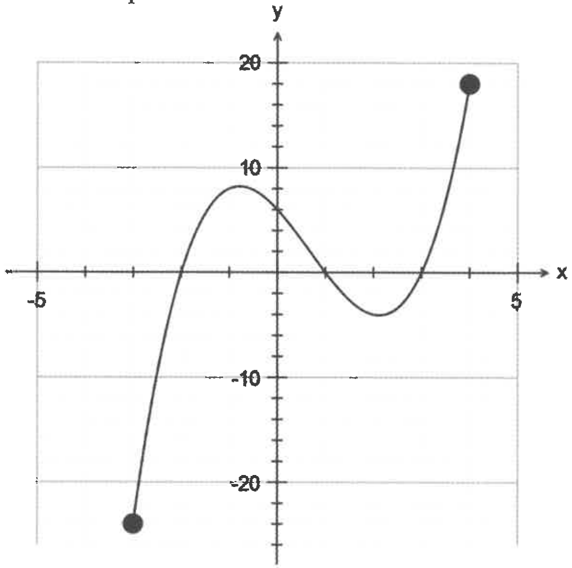
1. [4 marks: 1, 3]

A curve has equation $y = -(x-1)(x+2)(3-x)$.

(a) Find y when $x = -3$ and $x = 4$.

When $x = -3$, $y = -24$.
When $x = 4$, $y = 18$. ✓

(b) Find the intercepts of this curve and sketch this curve for $-3 \le x \le 4$.



On the Sketch:

y-intercept: $x = 0 \Rightarrow y = 6 \Rightarrow (0, 6)$. ✓
x-intercept: $y = 0 \Rightarrow x = -2, 1, 3 \Rightarrow (-2, 0), (1, 0) \& (3, 0)$. ✓

Correct End Points: $(-3, -24) \& (4, 18)$ ✓

Calculator Free

2. [7 marks: 4, 3]

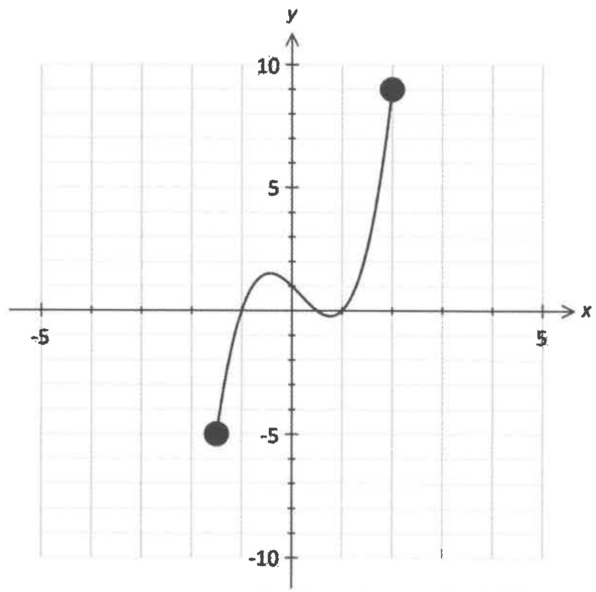
A curve has equation $y = 2x^3 - x^2 - 2x + 1$.

(a) Find the coordinates of the x-intercepts of this curve.

When $x = 1$, $y = 2 - 1 - 2 + 1 = 0$.
Hence, $(x - 1)$ is a factor.
$2x^3 - x^2 - 2x + 1 \equiv (x-1)(2x^2 + x - 1)$ ✓
$\equiv (x-1)(2x-1)(x+1)$ ✓

Hence, x-intercepts are:
$(-1, 0), (\frac{1}{2}, 0) \& (1, 0)$. ✓✓

(b) Sketch this curve for $-1.5 \le x \le 2$.



y-intercept: $x = 0 \Rightarrow y = 1 \Rightarrow (0, 1)$. ✓
x-intercept: $y = 0 \Rightarrow (-1, 0), (\frac{1}{2}, 0) \& (1, 0)$. ✓

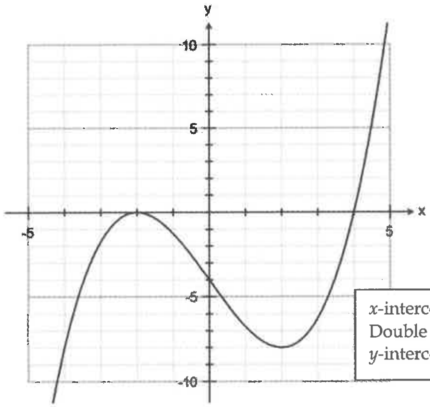
Correct End Points: $(-1.5, -5) \& (2, 9)$ ✓

Calculator Free

3. [6 marks: 3, 3]

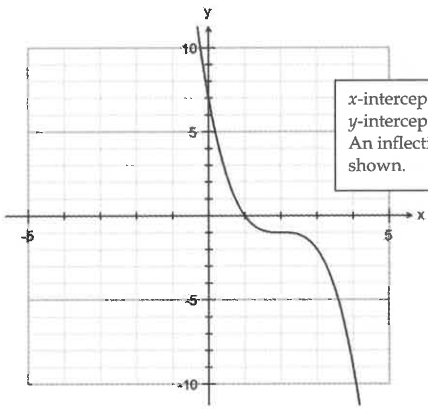
Give a possible sketch for each of the following cubic curves:

(a) A cubic with roots $x = -2, 4$ and y-intercept $(0, -5)$.



x-intercept at $x = 4$	✓
Double intercept at $x = -2$	✓
y-intercept	✓

(b) A cubic with root $x = 1$ and y-intercept $(0, 7)$.



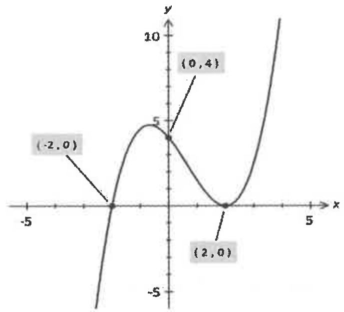
x-intercept at $x = 1$	✓
y-intercept	✓
An inflection point clearly shown.	✓

Calculator Free

4. [9 marks: 3, 3, 3]

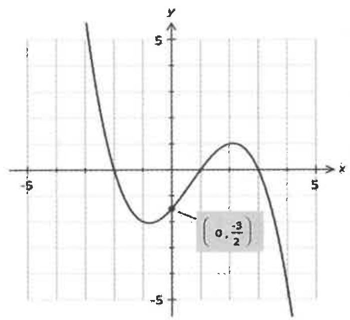
Determine the equations of each of the following cubic curves.

(a)



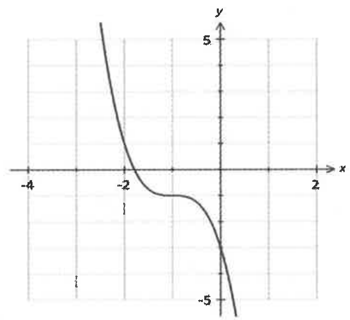
$y = k(x+2)(x-2)^2$ ✓✓
When $x = 0, y = 4 \Rightarrow k = 0.5$
Hence, $y = 0.5(x+2)(x-2)^2$ ✓

(b)



$y = k(x+2)(x-1)(x-3)$ ✓✓
When $x = 0, y = -1.5 \Rightarrow k = -0.25$
Hence, $y = -0.25(x+2)(x-1)(x-3)$ ✓

(c)



$y = k(x+1)^3 - 1$ ✓✓
When $x = 0, y = -3 \Rightarrow k = -2$
Hence, $y = -2(x+1)^3 - 1$ ✓

Calculator Free

5. [7 marks: 1, 2, 2, 2]

Equations of cubic curves can be written in the form $y = k(x-a)(x-b)(x-c)$ or $y = k(x-a)(x-b)^2$ or $y = ax^3 + bx^2 + cx + d$. Find a possible equation of a cubic curve if this curve has:

(a) exactly three roots $x = 1, 2, -1$.

> $y = k(x-1)(x-2)(x+1), \quad k \neq 0$ ✓

(b) exactly three roots $x = 1, 2, -1$ and y-intercept at $(0, -6)$

> $y = k(x-1)(x-2)(x+1)$
> When $x = 0, y = -6, \Rightarrow -6 = k(2) \Rightarrow k = -3$ ✓
> Hence, $y = -3(x-1)(x-2)(x+1)$ ✓

(c) exactly two roots $x = -1, 1$ and y-intercept at $(0, -6)$.

> $y = -6(x-1)^2(x+1)$ ✓✓
> or $y = 6(x-1)(x+1)^2$

(d) has exactly one root $x = 1$ and y-intercept $(0, 2)$

> $y = -2(x-1)^3$ or equivalent ✓✓

6. [6 marks: 1, 1, 2, 1, 1]

Consider the cubic curves :

I	$y = (x-1)(x+2)^2$	II	$y = (x+1)(x^2-1)$
III	$y = (x-1)^3 + 1$	IV	$y = (x+1)(1-x)(x+3)$

(a) Which of these curves have negative y-intercepts?

> 1 and II (both must be chosen) ✓

(b) Which of the given curves has three distinct (different) roots?

> II and IV (both must be chosen) ✓

Calculator Free

6. (c) Which of the given curves has two turning points?

> I, II and IV ✓✓ [−1 mark per error, omission]

(d) Which of the given curves has one turning point?

> None. ✓

(e) Which of the given curves has no turning point?

> III ✓

7. [6 marks: 2, 4]

Find all possible equations of a cubic curve with:

(a) roots $x = 1, 2, -3$ and vertical intercept $(0, 12)$.

> $y = k(x+3)(x-1)(x-2)$ ✓
> $x = 0, y = 12 \Rightarrow k = 2$
> Hence, $y = 2(x+3)(x-1)(x-2)$ ✓

(b) exactly two roots at $x = -2$ and $x = 4$ and vertical intercept $(0, 16)$

> $y = k(x+2)^2(x-4)$ ✓
> $x = 0, y = 16 \Rightarrow k = -1$
> Hence, $y = -(x+2)^2(x-4)$ ✓
>
> AND
> $y = k(x-4)^2(x+2)$ ✓
> $x = 0, y = 16 \Rightarrow k = \frac{1}{2}$
> Hence, $y = \frac{1}{2}(x-4)^2(x+2)$ ✓

Calculator Free

8. [12 marks: 2, 3, 2, 5]

Consider the cubic equation $y = f(x) = k(x+2)(x^2 - 3x + c)$ where k and c are constants.

(a) Find the value of c if $f(4) = f(-2) = f(-1) = 0$.

$f(x) = k(x-4)(x+2)(x+1)$
Hence, $x^2 - 3x + c = (x-4)(x+1)$ ✓
$= x^2 - 3x - 4$
Therefore, $c = -4$ ✓

(b) Find the value(s) of c if the cubic curve has three roots.

For the cubic to have three roots, the quadratic factor must have 2 real roots. ✓
Hence, the discriminant
$(-3)^2 - 4(1)(c) > 0$ ✓
$c < \frac{9}{4}$ ✓

(c) Find the value(s) of c if the cubic has exactly two roots.

For the cubic to have two roots, the quadratic factor must have one root. ✓
Hence, the discriminant
$(-3)^2 - 4(1)(c) = 0$
$c = \frac{9}{4}$ ✓

(d) Find the values of k and c if $f(-4) = f(-2) = 0$ and $f(0) = -4$.

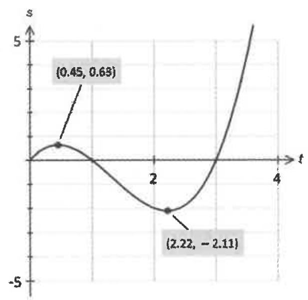
$f(x) = k(x+2)(x+4)(x+a)$ ✓
Hence, $x^2 - 3x + c = (x+4)(x+a)$ ✓
$= x^2 + (4+a)x + 4a$
Therefore, $4 + a = -3$ ✓
$a = -7$
$\Rightarrow c = 4a = -28$ ✓

$f(0) = -4 \Rightarrow k(2)(4)(-7) = -4$
$k = \frac{1}{14}$ ✓

Calculator Assumed

9. [8 marks: 2, 1, 2, 3]

The displacement, s metres, t seconds after a particle passes a fixed point O, is given by $s = t^3 - 4t^2 + 3t$, for $0 \le t \le 4$. The graph of s against t is given below. The graph has turning points at (0.45, 0.63) and (2.22, −2.11).



(a) Find when the particle returns to O.

$s = 0 \Rightarrow t = 1$ or 3 seconds ✓✓

(b) Find the displacement of the particle when $t = 2$.

$t = 2,\ s = -2$ m ✓

(c) Find the farthest distance out from O reached by the particle in the interval $0 \le t \le 1$.

0.63 m (corresponds to the turning point in the interval $0 \le t \le 1$) ✓✓

(d) Find the distance travelled by the particle in the first 2 seconds.

Changed direction at $t = 0.63$ ✓
s(0) = 0   s(1) = 0.63   s(2) = −2 ✓
Distance = 2 × 0.63 + 2 = 3.26 m ✓

Calculator Assumed

10. [9 marks: 1, 2, 2, 4]

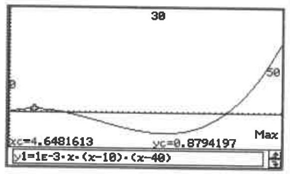
A particle P moves along a straight line. Its displacement t seconds after passing a fixed point O is given by $s = 0.001t(t-10)(t-40)$ metres, for $0 \le t \le 50$ seconds. Graph s against t on your graphic calculator. Use an appropriate routine to find:

(a) the displacement of P when $t = 50$ seconds.

When $t = 50$, $s = 0.001(50)(50-10)(50-40)$
$= 20$ m ✓

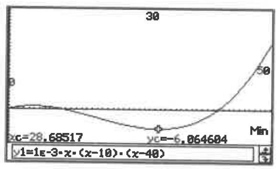
(b) the farthest P is from O for $0 \le t \le 10$ seconds.

Farthest distance = 0.88 m ✓✓
[corresponds to max. point at (4.65, 0.88) for $0 \le t \le 10$]

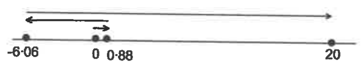


(c) the farthest P is from O for $10 \le t \le 40$ s

Farthest distance = 6.06 m ✓✓
[corresponds to min point at (28.69, –6.06) for $10 \le t \le 40$]



(d) the total distance travelled in the first 50 seconds.



Changed direction at $t = 4.65$ and $t = 28.69$. ✓✓
$s(0) = 0$ $s(4.65) = 0.88$ $s(28.69) = 6.06$ $s(50) = 20$

Total distance travelled $= 2 \times 0.88 + 2 \times 6.06 + 20$ ✓
$= 33.88$ m ✓

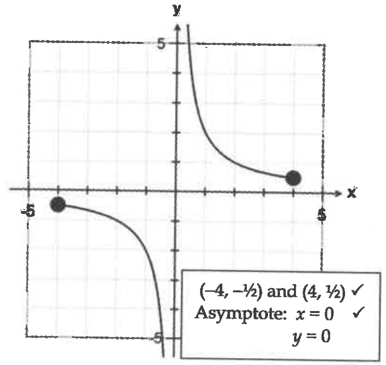
04 Rectangular Hyperbolas

Calculator Free

1. [4 marks: 2, 2]

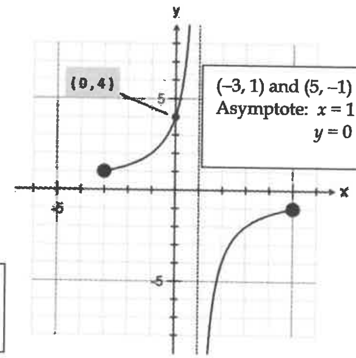
Sketch in the axes provided, the graph of y against x. Show clearly all intercepts (if any) and asymptotes (if any).

(a) $y = \dfrac{2}{x}$ for $-4 \le x \le 4$.



(–4, –½) and (4, ½) ✓
Asymptote: $x = 0$ ✓
$y = 0$

(b) $y = -\dfrac{4}{x-1}$ for $-3 \le x \le 5$



(–3, 1) and (5, –1) ✓
Asymptote: $x = 1$ ✓
$y = 0$

2. [6 marks]

Complete the table below for the following curves. –1 per error

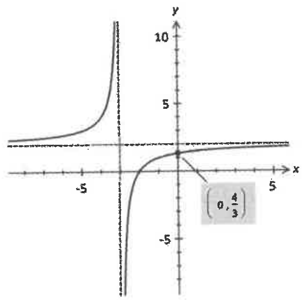
Curve	x-intercept	y-intercept	Asymptote parallel to the x-axis	Asymptote parallel to the y-axis
$y = \dfrac{4}{x-3}$	n/a	(0, –4/3)	$y = 0$	$x = 3$
$y = \dfrac{-3}{2x+9}$	n/a	(0, –1/3)	$y = 0$	$x = -9/2$
$y = \dfrac{5}{2-x}$	n/a	(0, 5/2)	$y = 0$	$x = 2$
$y = \dfrac{15}{x+5} - 3$	(0, 0)	(0, 0)	$y = -3$	$x = -5$
$(x+2)\,y = 10$	n/a	(0, 5)	$y = 0$	$x = -2$
$(x-1)\,(y-2) = 10$	(–4, 0)	(0, –8)	$y = 2$	$x = 1$

Calculator Free

3. [9 marks: 3, 3, 3]

Find the equation of each of the following rectangular hyperbola.

(a)



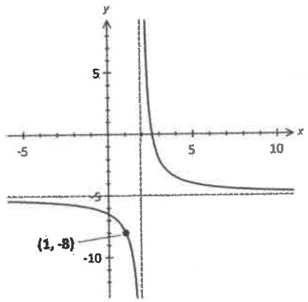
$y = \frac{k}{x+3} + 2$ ✓✓

When $x = 0$, $y = \frac{4}{3}$

$\frac{4}{3} = \frac{k}{3} + 2 \Rightarrow k = -2$

Hence, $y = \frac{-2}{x+3} + 2$ ✓

(b)



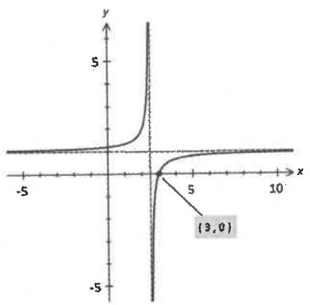
$y = \frac{k}{x-2} - 5$ ✓✓

When $x = 1$, $y = -8$

$-8 = -k - 5 \Rightarrow k = 3$

Hence, $y = \frac{3}{x-2} - 5$ ✓

(c)



$y = \frac{k}{x-2.5} + 1$ ✓✓

When $x = 3$, $y = 0$

$0 = \frac{k}{0.5} + 1 \Rightarrow k = -0.5$

Hence, $y = \frac{-0.5}{x-2.5} + 1$ ✓

or $y = \frac{-1}{2x-5} + 1$

Calculator Free

4. [3 marks: 1, 1, 1]

Consider the following curves:

I $y = -\frac{1}{x}$ II $y = \frac{2}{x}$ III $y = \frac{1}{2x}$ IV $xy = -4$

(a) Which of the given curves passes through the point (–1, –2)?

II ✓

(b) Which of the given curves has the property that when x is positive, y is negative?

I & IV ✓

(c) Which of the given curves has the property that as the value of x increases, the value of y decreases?

II & III ✓

---

5. [8 marks: 2, 2, 2, 2]

A rectangular hyperbola has asymptotes with equation $x = -2$ and $y = 4$.

(a) Write two possible equations for this function.

$y = \pm\frac{1}{x+2} + 4$ ✓✓

(b) Write the equation of this function if it has a y-intercept at (0, 5).

When $x = 0$, $y = 5$. Hence $y = \frac{2}{x+2} + 4$ ✓✓

(c) Write the equation of this function if it has a x-intercept at (–3, 0).

When $x = -3$, $y = 0$. Hence $y = \frac{4}{x+2} + 4$ ✓✓

(d) Write the equation of this function if it passes through the point (3, 5).

When $x = 3$, $y = 5$. Hence $y = \frac{5}{x+2} + 4$ ✓✓

Calculator Free

6. [8 marks: 2, 2, 2, 2]

Find y in terms of x if:

(a) y is inversely proportional to $2x-5$ and $y = 8$ when $x = 4$.

> Let $y = \frac{k}{2x-5}$
> When $x = 4, y = 8 \Rightarrow k = 24$. ✓
> Hence $y = \frac{24}{2x-5}$. ✓

(b) y is directly proportional to $\frac{1}{x}$ and $y = 5$ when $x = 20$.

> Let $y = \frac{k}{x}$
> When $x = 20, y = 5 \Rightarrow k = 100$. ✓
> Hence $y = \frac{100}{x}$. ✓

(c) y is directly proportional to $\frac{1}{x+6}$ and $y = -2$ when $x = 4$.

> Let $y = \frac{k}{x+6}$
> When $x = 4, y = -2 \Rightarrow k = -20$. ✓
> Hence $y = \frac{-20}{x+6}$. ✓

(d) y is inversely proportional to x^3 and $y = 80$ when $x = 2$.

> Let $y = \frac{k}{x^3}$
> When $x = 2, y = 80 \Rightarrow k = 640$. ✓
> Hence $y = \frac{640}{x^3}$. ✓

Calculator Free

7. [4 marks: 3, 1]

P is directly proportional to x and inversely proportional to y.
If P = 5 when $x = 1$ and $y = 4$, find:
(a) P in terms of x and y.

> Let $P = \frac{kx}{y}$. ✓
> When $x = 1, y = 4$ and $P = 5 \Rightarrow k = 20$. ✓
> Hence, $P = \frac{20x}{y}$. ✓

(b) y when P = 100 and $x = 20$.

> $y = 4$ ✓

8. [5 marks: 3, 2]

P is directly proportional to x^2 and inversely proportional to y^3.
If P = 20 when $x = 4$ and $y = 2$, find:
(a) P in terms of x and y.

> Let $P = \frac{kx^2}{y^3}$. ✓
> When $x = 4, y = 2$ and $P = 20 \Rightarrow k = 10$. ✓
> Hence, $P = \frac{10x^2}{y^3}$ ✓

(b) x when P = 2 and $y = 5$.

> $x = \pm 5$ ✓✓

9. [3 marks]

T varies directly with the square root of I and varies inversely with the square root of M. Given that T = 2π, when I = 9 and M = 4. Find T in terms of I and M.

> Let $T = k\sqrt{\frac{I}{M}}$. ✓
> When I = 9, M = 4 and T = $2\pi \Rightarrow k = 4\pi/3$. ✓
> Hence, $T = \frac{4\pi}{3}\sqrt{\frac{I}{M}}$. ✓

Calculator Assumed

10. [3 marks: 1, 1, 1]

A task can be completed by 8 workers in 6 days.

(a) How many workers would be required to complete the same task in half the time?

> 16 workers ✓

(b) How many days would double the number of workers take to complete the same task?

> 3 days ✓

(c) What is the minimum number of workers required to complete the task in 12 days?

> 4 workers ✓

---

11. [3 marks: 2, 1]

At constant temperature, the volume of a gas (V m^3) varies inversely as its pressure (P pascals). If at a fixed temperature, 100 m^3 of a gas exerts a pressure of 200 pascals, find the:

(a) pressure exerted when the volume is 500 m^3.

> PV = 20 000. Hence, when V = 500, P = 40 pascals ✓✓

(b) volume of the gas if the pressure exerted is 20 pascals.

> PV = 20 000. Hence, when P = 20, V = 1 000 m^3 ✓

---

12. [3 marks: 2, 1]

The amount of current (amps) flowing through an electrical circuit is inversely proportional to its resistance (ohms). If the current flow is 5 amps when the resistance is 8 ohms, find the:

(a) current flow when the resistance is 0.5 ohms.

> IR = 40. Hence, when R = 0.5, I = 80 amps ✓✓

(b) resistance when the current flow is 15 amps.

> IR = 40. Hence, when I = 15, R = 8/3 ohms ✓

Calculator Assumed

13. [4 marks]

A food drop can feed 24 hikers for 6 whole days. Assuming the daily rations per hiker remains constant and given that there were at least 6 hikers, what are the possible numbers of hikers if the food is to last at least 10 whole days. Justify your answer.

> Let x: No. of hikers and y: No. of days
> $\Rightarrow xy = 144$ ✓
>
> When $y = 10, x = 14.4$ ✓
> For $y \geq 10, x \leq 14$. ✓
>
> Hence, 14 hikers or less. ✓

---

14. [6 marks]

A project can be completed by 18 workers in 5 weeks. The same task is to be completed in exactly a whole number of weeks. How many workers would be required to achieve this? State all the possible combinations.

$xy = 90$ ✓

x workers	y weeks
1	90
2	45
3	30
5	18
6	15
10	9
15	6
18	5
30	3
45	2
90	1

−1 per error ✓✓✓✓✓

05 Exponential Functions I

Calculator Free

1. [7 marks: 1, 1, 2, 3]

Consider the curve with equation $y = 2^x - 4$.

(a) State the equation of the horizontal asymptote of this curve.

$y = -4$ ✓

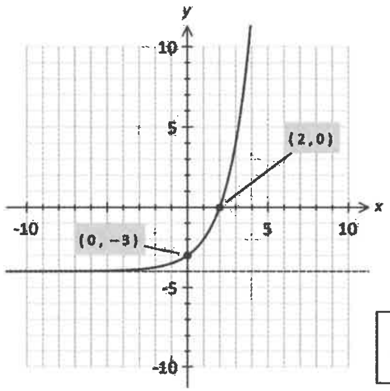
(b) Find the coordinates of the vertical intercept of this curve.

$x = 0 \Rightarrow y = -3$
Hence $(0, -3)$. ✓

(c) Find the coordinates of the horizontal intercept of this curve.

$y = 0 \Rightarrow 2^x - 4 = 0$
$x = 2$ ✓
Hence $(2, 0)$. ✓

(d) On the axes provided below, sketch this curve.
Indicate clearly the intercepts and the asymptote(s).



Asymptote ✓
Intercepts ✓✓

Calculator Free

2. [7 marks: 1, 1, 2, 3]

Consider $y = 4 - 3^x$

(a) State the equation of the horizontal asymptote of this curve.

$y = 4$ ✓

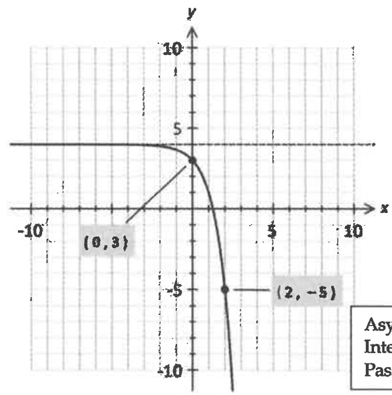
(b) Find the coordinates of the vertical intercept of this curve.

$x = 0 \Rightarrow y = 3$.
Hence $(0, 3)$ ✓

(c) Find the point of intersection between this curve and the line $y = -5$.

$4 - 3^x = -5$
$x = 2$ ✓
Hence $(2, -5)$. ✓

(d) On the axes provided below, sketch this curve.



Asymptote ✓
Intercept ✓
Passes through $(2, -5)$ ✓

Calculator Free

3. [4 marks: 2, 2]

(a) State two possible equations for an exponential curve with asymptote $y = -2$ and vertical intercept $(0, -1)$.

> Any two equations of the form $y = -2 + a^x$ for $a > 0$. ✓✓

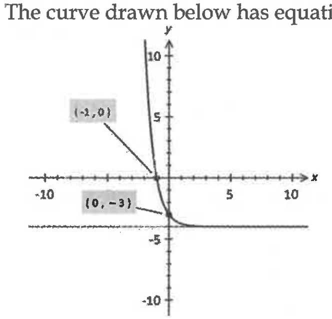
(b) State two possible equations for an exponential curve with asymptote $y = 2$ and vertical intercept $(0, -3)$.

> Any two equations of the form $y = 2 - 5a^x$ for $a > 0$. ✓✓

---

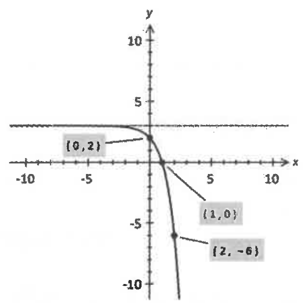
4. [7 marks: 2, 5]

(a) The curve drawn below has equation of the form $y = a^{-x} + b$. Find a and b.



> Horizontal asymptote is $y = -4$
> Hence, $b = -4$. ✓
> $y = a^{-x} - 4$
> $x = -1, y = 0 \Rightarrow a = 4$ ✓

(b) The curve drawn below has equation of the form $y = ka^x + b$. Find a, b and k.



> Horizontal asymptote is $y = 3$
> Hence, $b = 3$. ✓
> $y = ka^x + 3$
> $x = 1, y = 0 \Rightarrow ka = -3$ ✓
> $x = 2, y = -6 \Rightarrow ka^2 = -9$ ✓
> $a = 3$ ✓
> $k = -1$ ✓

06 Square Root Functions

Calculator Free

1. [9 marks: 2, 2, 2, 3]

Consider the curve with equation $y = \sqrt{x-4}$.

(a) Explain why it is not possible for this curve to exist for values of $x < 4$.

> For $x < 4$, $y = \sqrt{negative\ \ number}$. ✓
> $\sqrt{negative\ \ number}$ is not a real number. ✓
> Hence, curve does not exist for $x < 4$.

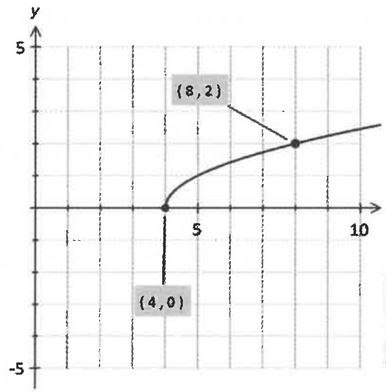
(b) Find the coordinates of the horizontal intercept of this curve.

> $\sqrt{x-4} = 0 \Rightarrow x = 4$. ✓
> Hence, (4, 0). ✓

(c) Determine the point of intersection between this curve and the line $y = 2$.

> When $y = 2$, $\sqrt{x-4} = 2$
> $x - 4 = 4$ ✓
> $x = 8$
> Hence, (8, 2). ✓

(d) On the axes provided below, sketch this curve.



> Starts at (4, 0) ✓
> Passes (8, 2) ✓
> Correct shape. ✓

Calculator Free

2. [9 marks: 2, 2, 2, 3]

Consider the curve with equation $y = 2 + \sqrt{x+9}$.

(a) Explain why $y \geq 2$.

> $y = 2$ + the square root of a number.
> The square root of a number is always non-negative. ✓
> Hence, $y = 2$ + a non-negative number. ✓
> $\Rightarrow$ $y \geq 2$.

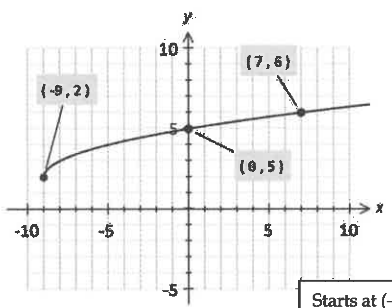
(b) Find the coordinates of the vertical intercept of this curve.

> When $x = 0, y = 2 + \sqrt{9} = 5$ ✓
> Hence, (0, 5). ✓

(c) Determine the point of intersection between this curve and the line $y = 6$.

> When $y = 6$, $2 + \sqrt{x+9} = 6$
> $\sqrt{x+9} = 4$ ✓
> $x + 9 = 16 \Rightarrow x = 7$
> Hence, (7, 6). ✓

(d) On the axes provided below, sketch this curve.



> Starts at (−9, 2) ✓
> Passes (0, 5) & (7, 6) ✓
> Correct shape. ✓

Calculator Free

3. [4 marks: 2, 2]

(a) State two possible equations for the curve with equation $y = a + k\sqrt{x+b}$ if the curve has $x \leq 2$ and $y \leq -3$.

> Any two equations of the form $y = -3 + k\sqrt{2-x}$ for $k < 0$. ✓✓

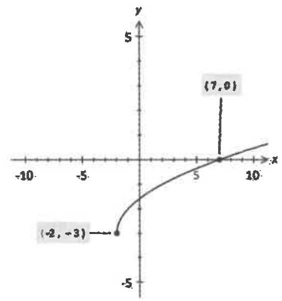
(b) State two possible equations for the curve with equation $y = a + k\sqrt{x+b}$ if the curve has $x \geq -3$ and $y \geq 5$.

> Any two equations of the form $y = 5 + k\sqrt{x+3}$ for $k > 0$. ✓✓

---

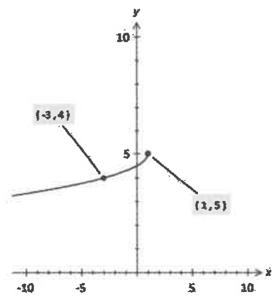
4. [6 marks: 3, 3]

(a) Find the equation of the curve drawn below with equation $y = a + k\sqrt{x+b}$.



> $y = -3 + k\sqrt{x+2}$ ✓✓
> $x = 7, y = 0 \Rightarrow k = 1$ ✓
> Hence $y = -3 + \sqrt{x+2}$.

(b) Find the equation of the curve drawn below with equation $y = a + k\sqrt{x+b}$.



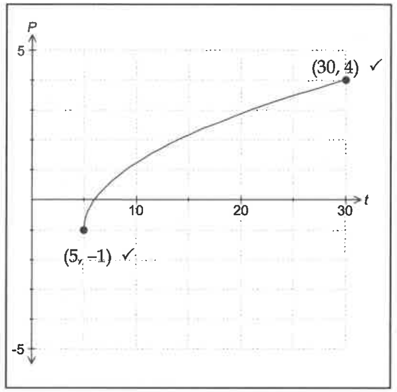
> $y = 5 + k\sqrt{1-x}$ ✓✓
> $x = -3, y = 4 \Rightarrow 4 = 5 + 2k$
> $k = -0.5$ ✓
> Hence $y = 5 - 0.5\sqrt{1-x}$.

Calculator Assumed

5. [8 marks: 2, 2, 2, 2]

The daily Profit (in hundreds of dollars) for a small Lunch Bar is modelled by $P = -1 + \sqrt{t-5}$ for $5 \le t \le 30$, where t is time in days after 1st July.

(a) Sketch P against t in the axes provided below. Show clearly all essential features of the graph.



(b) On what date did the Lunch Bar open for Business and what was the profit for that day?

6th July ✓
Loss of $100 ✓

(c) How many days did the Lunch Bar take to make its first profit?

2 days ✓

(d) What was the profit, three weeks after the Lunch Bar first opened.

P(21) = 3 ✓
Profit = $300 ✓

07 Circles & Parabolas

Calculator Free

1. [9 marks: 2, 2, 2, 3]

Consider the circle with equation $(x-1)^2 + (y-3)^2 = 10$.

(a) Find the coordinates of the x-intercepts.

$y = 0 \Rightarrow (x-1)^2 + (-3)^2 = 10$
$(x-1)^2 = 1$ ✓
$x = 0, 2$
Hence, (0, 0) & (2, 0). ✓

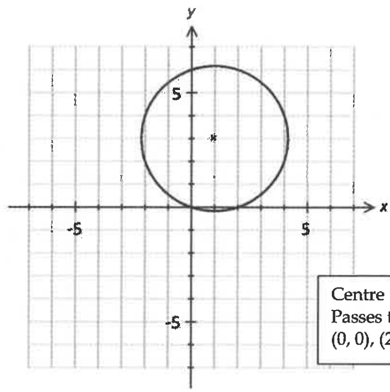
(b) Find the coordinates of the y-intercepts.

$x = 0 \Rightarrow (-1)^2 + (y-3)^2 = 10$
$(y-3)^2 = 9$ ✓
$y = 0, 6$
Hence, (0, 0) & (0, 6). ✓

(c) Determine the coordinates of the centre of this circle and its radius.

Centre (1, 3). ✓
Radius = $\sqrt{10}$ ✓

(d) On the axes provided below, sketch this circle.



Centre (1, 3). ✓
Passes through:
(0, 0), (2, 0) & (0, 6) ✓✓

Calculator Free

2. [10 marks: 2, 3, 2, 3]

Consider the circle with equation $(x+2)^2+(y+3)^2=25$.

(a) Find the coordinates of the x-intercepts.

> $y=0 \Rightarrow (x+2)^2+(3)^2=25$
> $(x+2)^2=16$ ✓
> $x=-6, 2$
> Hence, $(-6, 0)$ & $(2, 0)$. ✓

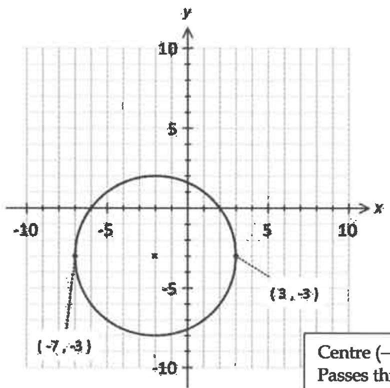
(b) Find the coordinates of the y-intercepts.

> $x=0 \Rightarrow (2)^2+(y+3)^2=25$
> $(y+3)^2=21$ ✓
> $y=-3\pm\sqrt{21}$
> Hence, $(0, -3-\sqrt{21})$ & $(0, -3+\sqrt{21})$. ✓✓

(c) Determine the coordinates of the centre of this circle and its radius.

> Centre $(-2, -3)$. ✓
> Radius = 5 ✓

(d) On the axes provided below, sketch this circle.



> Centre $(-2, -3)$. ✓
> Passes through:
> $(-6, 0)$, $(2, 0)$ ✓
> $(-7, -3)$, $(3, -3)$ ✓

Calculator Free

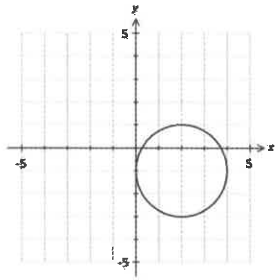
3. [4 marks]

State two possible equations for a circle with radius 5 and passing through the point with coordinates (4, 4).

> Any two circle equations with centre of circle (a, b)
> satisfying $(a-4)^2+(b-4)^2=25$.
> For example: $(x-9)^2+(y-4)^2=25$, $(x+1)^2+(y-4)^2=25$
> $(x-4)^2+(y-9)^2=25$, $(x-4)^2+(y+1)^2=25$
> For each equation: Centre ✓ Radius ✓

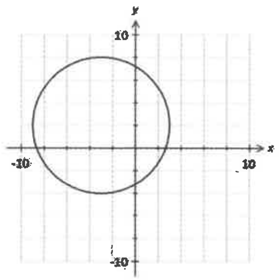
4. [6 marks: 3, 3]

(a) Find the equation of the circle drawn below.



> Circle is symmetrical about lines
> $x=2$ and $y=-1$.
> Hence, centre of circle is at $(2, -1)$.
> Diameter = 4. Hence radius = 2.
> Therefore $(x-2)^2+(y+1)^2=4$ ✓ ✓ ✓

(b) Find the equation of the circle drawn below.

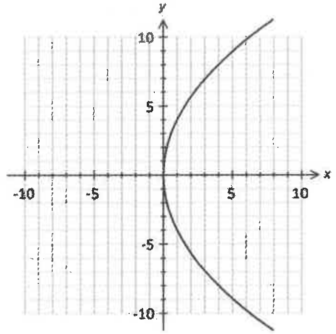


> Circle is symmetrical about lines
> $x=-3$ and $y=2$.
> Hence, centre of circle is at $(-3, 2)$.
> Diameter = 12. Hence radius = 6.
> Therefore $(x+3)^2+(y-2)^2=36$ ✓ ✓ ✓

Calculator Free

5. [3 marks]

On the axes provided, sketch the parabola with equation $y^2 = 16x$.



Parabolic curve with x-axis as axis of symmetry
passing through (0, 0) ✓
(1, 4), (1, –4) ✓
(4, 8) & (4, –8) ✓

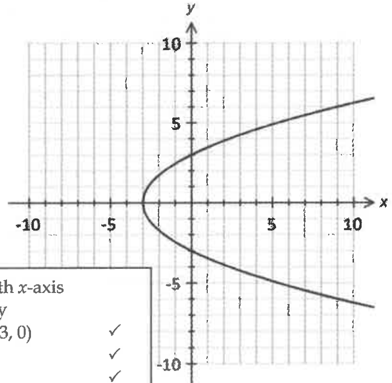
6. [6 marks: 3, 3]

Consider the parabola with equation $y^2 = 3(x + 3)$.

(a) State the coordinates of the x and y intercepts.

$x = 0, y^2 = 9 \Rightarrow y = \pm 3$
Hence (0, –3) & (0, 3). ✓✓
$y = 0 \Rightarrow x = -3$.
Hence (–3, 0). ✓

(b) On the axes provided sketch this parabola.



Parabolic curve with x-axis as axis of symmetry
passing through (–3, 0) ✓
(0, –3), (0, 3) ✓
(9, 6) & (9, –6) ✓

Calculator Free

7. [4 marks: 2, 2]

State a possible equation for a parabola passing through the point (1, 4):

(a) symmetrical about the y-axis.

Any equation of the form $y = kx^2 + c$ where $k + c = 4$.
For example: $y = kx^2$ ✓
When $x = 1, y = 4 \Rightarrow k = 4$
Hence, $y = 4x^2$ ✓

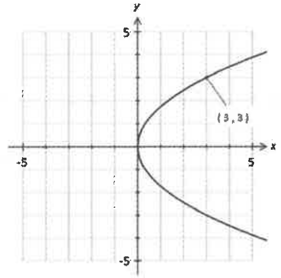
(b) symmetrical about the x-axis.

Any equation of the form $y^2 = kx + c$ where $k + c = 16$.
For example: $y^2 = kx$ ✓
When $x = 1, y = 4 \Rightarrow k = 16$
Hence, $y^2 = 16x$ ✓

8. [4 marks: 2, 2]

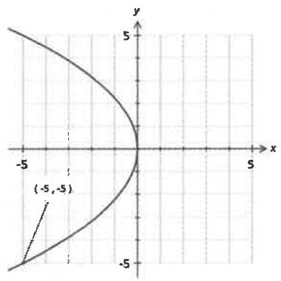
Find the equation of the parabola drawn below.

(a)



$y^2 = kx$ ✓
When $x = 3, y = 3 \Rightarrow k = 3$.
Hence $y^2 = 3x$ ✓

(b)



$y^2 = kx$ ✓
When $x = -5, y = 5 \Rightarrow k = -5$.
Hence $y^2 = -5x$ ✓

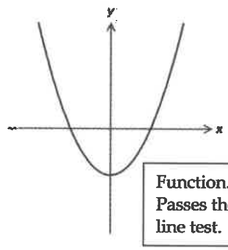
08 Functions & Relations I

Calculator Free

1. [6 marks: 1 each]

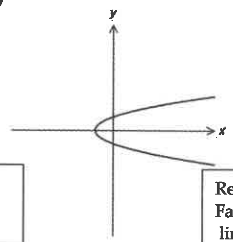
Determine with reasons if each of the following graphs represent functions or relations.

(a)



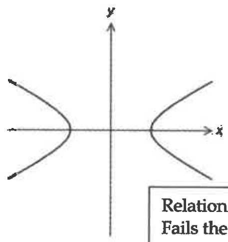
Function. Passes the vertical line test. ✓

(b)



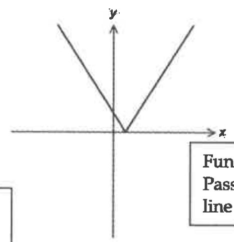
Relation. Fails the vertical line test. ✓

(c)



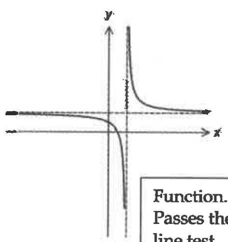
Relation. Fails the vertical line test. ✓

(d)



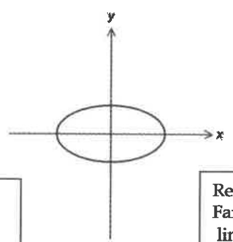
Function. Passes the vertical line test. ✓

(e)



Function. Passes the vertical line test. ✓

(f)



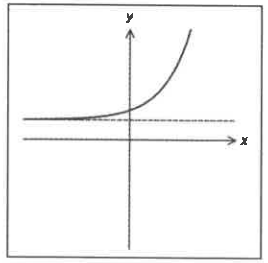
Relation. Fails the vertical line test. ✓

Calculator Free

2. [9 marks: 2, 2, 2, 3]

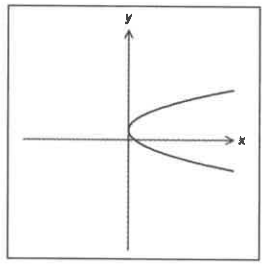
In the axes provided, make a sketch of:

(a) the graph of a function which has a horizontal asymptote.



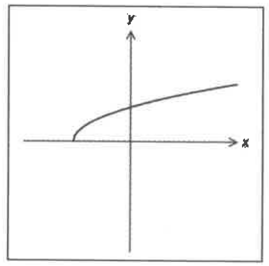
Horizontal asymptote present. ✓
Passes vertical line test. ✓

(b) the graph of a relation that is not symmetrical about the x-axis.



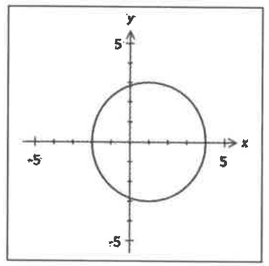
Not symmetrical about x-axis. ✓
Fails vertical line test. ✓

(c) the graph of a function that exists only for certain values of x.



Domain not **R**. ✓
Passes vertical line test. ✓

(d) the graph of a relation which is not symmetrical about the y-axis but symmetrical about the x-axis.

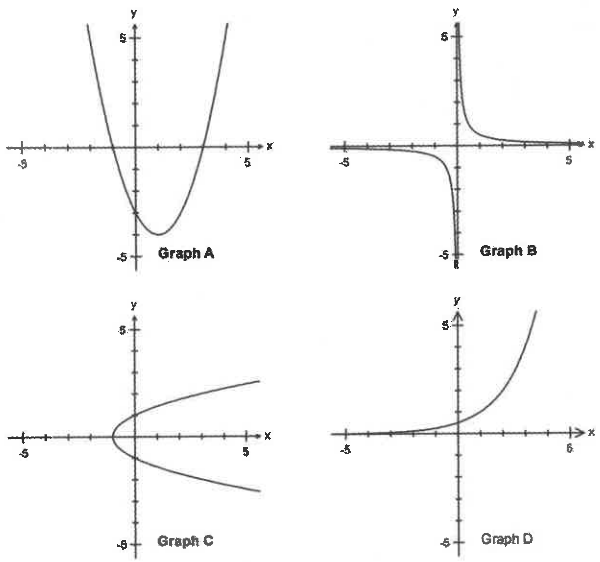


Scale given (essential). ✓
Symmetries satisfied. ✓
Fails vertical line test. ✓

Calculator Free

3. [4 marks: 1 each]

Match each of the following graphs with an equation from the given list.



Equation I: $y = \frac{1}{2x}$ — Equation II: $y = x^2 - 2x - 3$

Equation III: $y = 2^{x-1}$ — Equation IV: $y^2 = x - 1$

Equation V: $y = (x+1)^2 - 4$ — Equation VI: $x = y^2 - 1$

Equation VII: $y = 2^x$ — Equation VII: $y = \frac{1}{x}$

Graph	Equation
A	II $y = x^2 - 2x - 3$ ✓
B	I $y = \frac{1}{2x}$ ✓
C	VI $x = y^2 - 1$ ✓
D	III $y = 2^{x-1}$ ✓

Calculator Free

4. [5 marks]

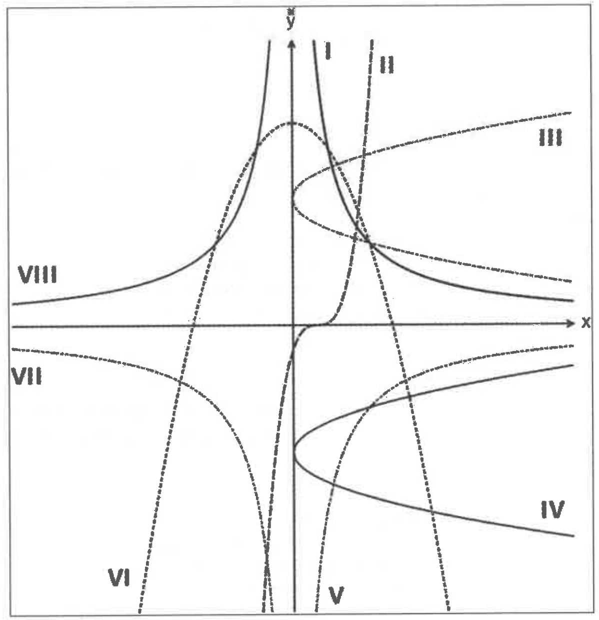
Match each of the following equations with one (or more) of the given curves.

Equation A: $y = \frac{1}{2}(16 - x^2)$

Equation B: $y = x^3 - 3x^2 + 3x - 1$

Equation C: $y = \frac{10}{x}$

Equation D: $x = (y+5)^2$

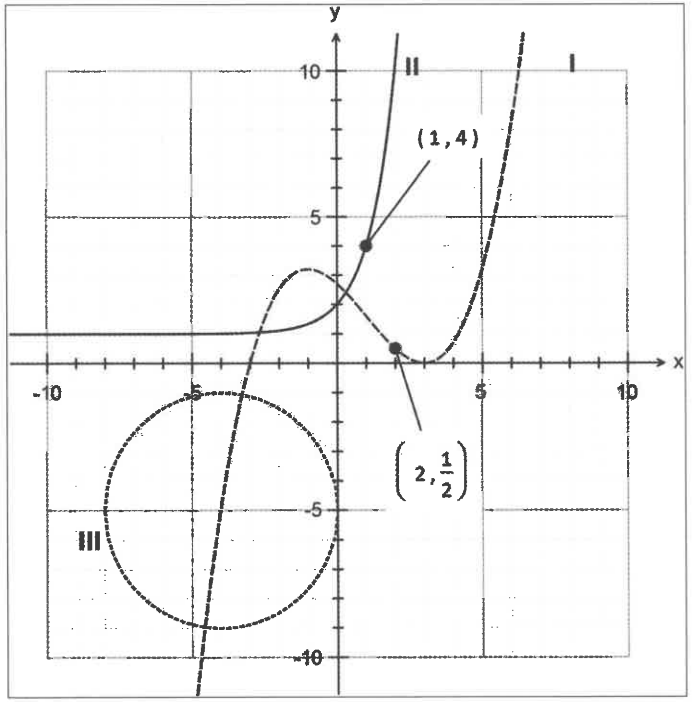


Equation	Graph
A	VI ✓
B	II ✓
C	I & VII ✓✓
D	IV ✓

Calculator Free

5. [8 marks]

Find the equation of the curves labelled I, II and III:

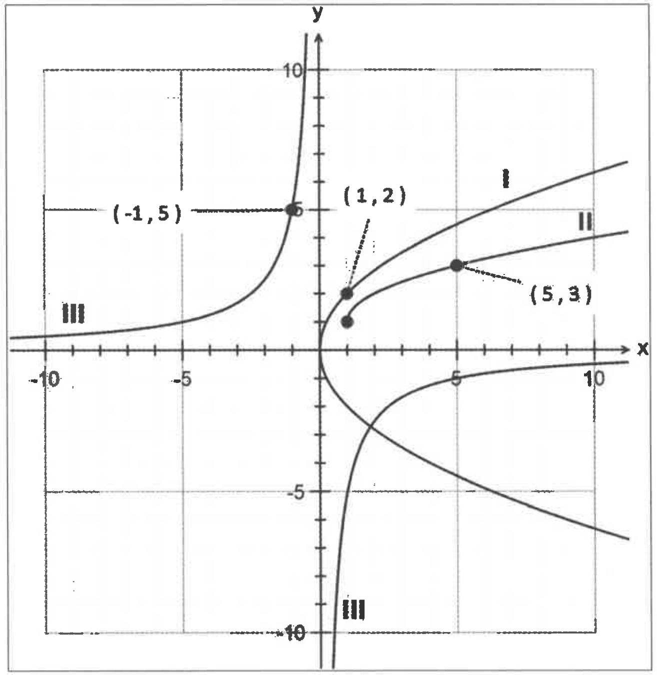


Curve	Equation
I	$y = 0.1(x+3)(x-3)^2$ ✓✓✓
II	$y = 3^x + 1$ ✓✓
III	$(x+4)^2 + (y+5)^2 = 16$ ✓✓✓

Calculator Free

6. [6 marks]

Find the equation of the curves labelled I, II and III:



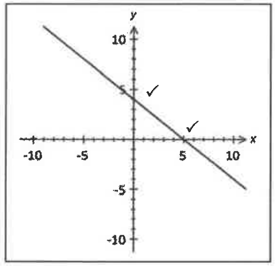
Curve	Equation
I	$y^2 = 4x$ ✓✓
II	$y = 1 + \sqrt{x-1}$ ✓✓
III	$y = -\frac{5}{x}$ ✓✓

Calculator Free

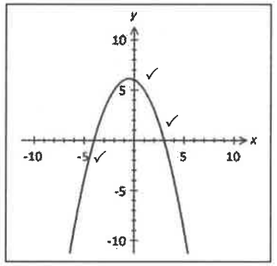
7. [15 marks: 2, 3, 2, 3, 2, 3]

Sketch each of the following. Indicate clearly all intercepts, asymptotes and symmetries where appropriate.

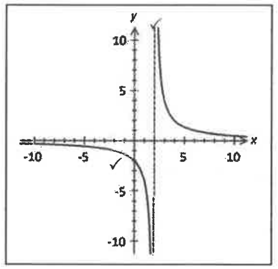
(a) $4x + 5y = 20$



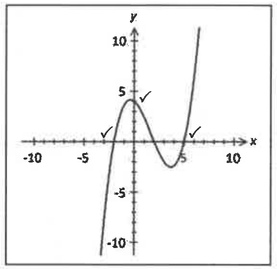
(b) $y = -0.5(x + 4)(x - 3)$



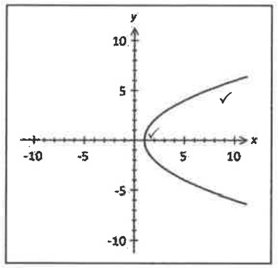
(c) $y = \dfrac{4}{x-2}$



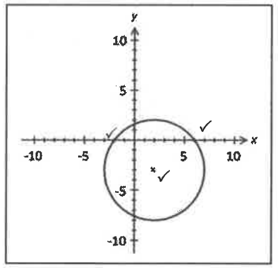
(d) $y = 0.2(x^2 - 4)(x - 5)$



(e) $y^2 = 4(x - 1)$



(f) $(x - 2)^2 + (y + 3)^2 = 25$

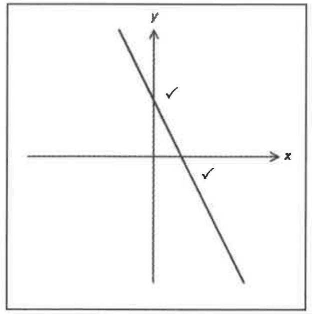


Calculator Free

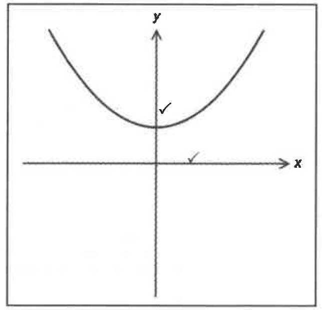
8. [6 marks: 2, 2, 2]

In the axes provided sketch:

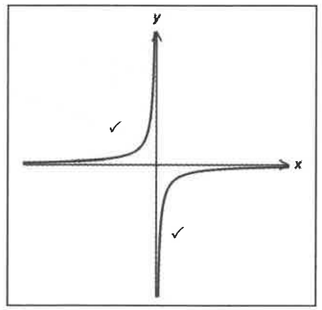
(a) a line with negative gradient and a positive y-intercept.



(b) a parabola with a positive y-intercept with no roots.



(c) a reciprocal function where the x and y values have opposite signs.

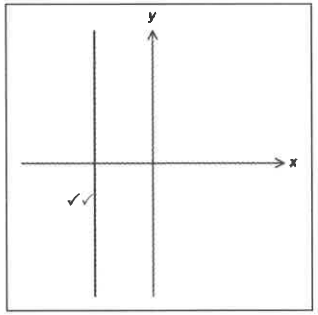


Calculator Free

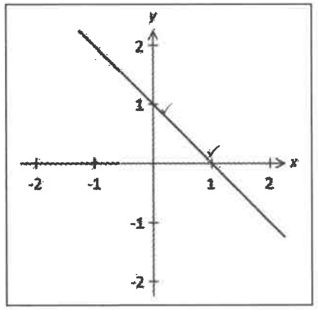
9. [12 marks: 4, 4, 4]

(a) Make a sketch of $ax + by = c$ where a, b and c are constants if:

(i) $a < 0$ and $b = 0$ and $c > 0$

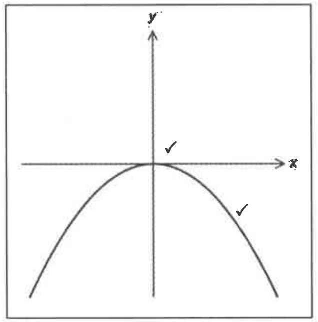


(ii) $a = b = c$

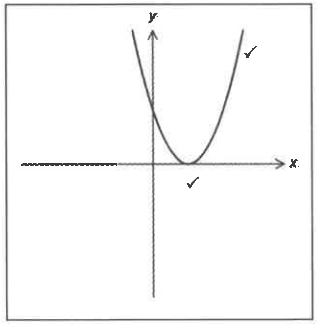


(b) Make a sketch of $y = ax^2 + bx + c$ where a, b and c are constants if:

(i) $a < 0$ and $b = 0$ and $c = 0$

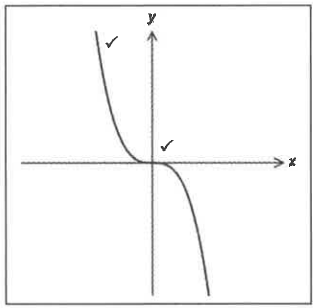


(ii) $a > 0$ and $b^2 = 4ac$

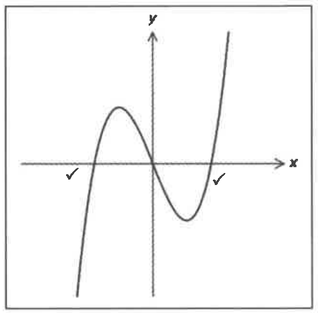


(c) Make a sketch of $y = k(x + a)(x + b)(x + c)$ where k, a, b and c are constants if:

(i) $k < 0$ and $a = b = c = 0$



(ii) $k > 0$ and $a = -b$ and $c = 0$

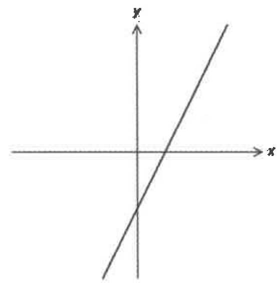


Mathematics Methods Revision Series Units 1 & 2 *08 Functions & Relations I*

Calculator Free

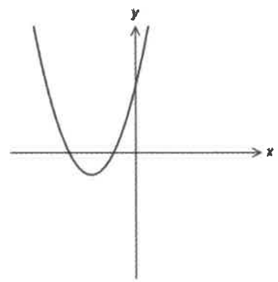
10. [7 marks: 2, 3, 2]

(a) The graph of $ax + by = c$ where a, b and c are constants is given in the accompanying diagram. If $a < 0$ and $b > 0$ determine with reasons if c is positive or negative.



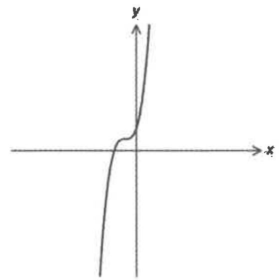
> y-intercept $\frac{c}{b} < 0$ ✓
> Since $b > 0$, $c < 0$ ✓

(b) The graph of $y = ax^2 + bx + c$ where a, b and c are constants is shown in the accompanying diagram. Explain clearly why $a > 0$, $b > 0$ and $c > 0$.



> Since parabola has a minimum point, $a > 0$. ✓
> Since y-intercept is positive, $c > 0$. ✓
> The line of symmetry $x = \frac{-b}{2a} < 0$.
> Since, $a > 0$, $b > 0$. ✓

(c) The graph of $y = k(x + m)(ax^2 + bx + c)$ where k, m, a, b and c are constants is shown in the accompanying diagram. Explain clearly why $b^2 - 4ac < 0$.



> The cubic function has one real root. ✓
> Hence, the quadratic factor must have no real roots. $\Rightarrow b^2 - 4ac < 0$ ✓

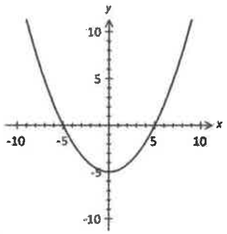
09 Functions & Relations II: Domain & Range

Calculator Free

1. [12 marks: 2, 2, 2, 2, 2, 2]

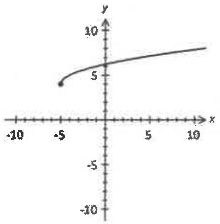
The graphs of several relations/functions are shown in the accompanying diagrams. In each case, state the domain and range for each relation/function.

(a)



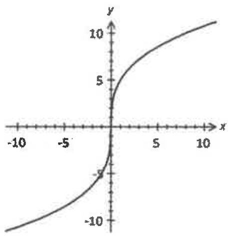
Domain: $\Re$ ✓
Range: $\{y: y \geq 5, y \in \Re\}$ ✓

(b)



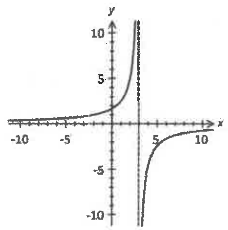
Domain: $\{x: x \geq -5, x \in \Re\}$ ✓
Range: $\{y: y \geq 4, y \in \Re\}$ ✓

(c)



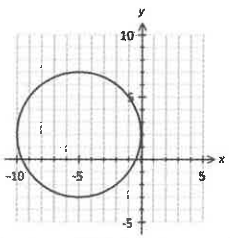
Domain: $\Re$ ✓
Range: $\Re$ ✓

(d)



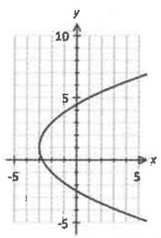
Domain: $\{x: x \neq 3, x \in \Re\}$ ✓
Range: $\{y: y \neq 0, y \in \Re\}$ ✓

(e)



Domain: $\{x: -10 \leq x \leq 0, x \in \Re\}$ ✓
Range: $\{y: -3 \leq y \leq 7, y \in \Re\}$ ✓

(f)



Domain: $\{x: x \geq -3, x \in \Re\}$ ✓
Range: $\Re$ ✓

Calculator Free

2. [20 marks: 1 each]

State the natural domain and range for each of the relations/functions below.

Function	Natural Domain	Natural Range
$y = (x+1)^2 - 5$	$\Re$ ✓	$\{y: y \geq -5, y \in \Re\}$ ✓
$y = 4 - 2(3x-1)^2$	$\Re$ ✓	$\{y: y \leq 4, y \in \Re\}$ ✓
$y = \sqrt{x-5}$	$\{x: x \geq 5, x \in \Re\}$ ✓	$\{y: y \geq 0, y \in \Re\}$ ✓
$y = \sqrt{x+3} - 10$	$\{x: x \geq -3, x \in \Re\}$ ✓	$\{y: y \geq -10, y \in \Re\}$ ✓
$y = 5^x + 3$	$\Re$ ✓	$\{y: y > 3, y \in \Re\}$ ✓
$y = -4 - 2^x$	$\Re$ ✓	$\{y: y < -4, y \in \Re\}$ ✓
$y = \dfrac{1}{x-1} + 3$	$\{x: x \neq 1, x \in \Re\}$ ✓	$\{y: y \neq 3, y \in \Re\}$ ✓
$y = 5 - \dfrac{3}{2x-4}$	$\{x: x \neq 2, x \in \Re\}$ ✓	$\{y: y \neq 5, y \in \Re\}$ ✓
$(x+1)^2 + (y+1)^2 = 4$	$\{x: -3 \leq x \leq 1, x \in \Re\}$ ✓	$\{y: -3 \leq y \leq 1, y \in \Re\}$ ✓
$y^2 = 4(x-1)$	$\{x: x \geq 1, x \in \Re\}$ ✓	$\Re$ ✓

Calculator Free

3. [12 marks: 3, 2, 2, 2, 3]

Consider the function with equation $y = \sqrt{16-x^2}$.

(a) The coordinates of the x-intercepts of this curve are $(a, 0)$ and $(b, 0)$ where $a \le b$. Find a and b.

> $y = 0 \Rightarrow 16 - x^2 = 0$ ✓
> $x = \pm 4$
> $a = -4$ & $b = 4$ ✓✓

(b) Explain why this curve only exists for values of x in the interval $a \le x \le b$.

> For values of x in the interval $-4 \le x \le 4$,
> $16 - x^2$ is always non-negative.
> Hence, $\sqrt{16-x^2}$ is possible. ✓
> For values of x outside this interval,
> $16 - x^2$ is always negative.
> Hence, $\sqrt{16-x^2}$ is not possible. ✓

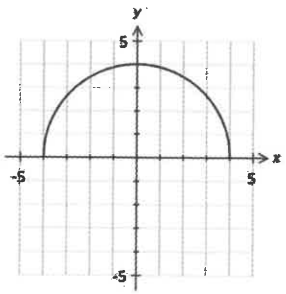
(c) What is the minimum and maximum value of y?

> y has a minimum value when $x = 4 \Rightarrow$ Minimum for $y = 0$ ✓
>
> y has a maximum value when $x = 0 \Rightarrow$ Maximum for $y = 4$ ✓

(d) Determine the domain and range of this function.

> Domain: $\{x: -4 \le x \le 4, x \in \Re\}$ ✓
> Range: $\{y: 0 \le y \le 4, y \in \Re\}$ ✓

(e) On the axes provided, sketch this curve.



> Semi-circle. ✓
> Centre (0, 0) ✓
> Radius = 4 ✓

Calculator Free

4. [10 marks: 2, 1, 2, 2, 3]

Consider the function with equation $y = 5 + \sqrt{9 - x_2}$.

(a) Explain why this curve exists only for $-3 \le x \le 3$.

> For values of x in the interval $-3 \le x \le 3$,
> $9 - x^2$ is always non-negative.
> Hence, $\sqrt{9-x^2}$ is possible. ✓
> For values of x outside this interval,
> $9 - x^2$ is always negative.
> Hence, $\sqrt{9-x^2}$ is not possible. ✓

(b) Explain why the y-value must always be at least 5.

> $y = 5$ + non-negative number. ✓
> Hence $y \ge 5$.

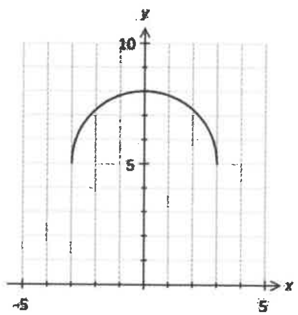
(c) What is the largest possible value of y?

> When $x = 0$, $\sqrt{9-x^2} = 3$ ✓
> Hence, largest possible value for $y = 5 + 3 = 8$. ✓

(d) Determine the domain and range of this function.

> Domain: $\{x: -3 \le x \le 3, x \in \Re\}$ ✓
> Range: $\{y: 5 \le y \le 8, y \in \Re\}$ ✓

(e) On the axes provided, sketch this curve.



> Semi-circle. ✓
> Centre (0, 5) ✓
> Radius = 3 ✓

Calculator Assumed

5. [9 marks: 3, 3, 3]

Consider the function $f(x) = x + 1$.

(a) Express in terms of x, $y = f(x^3)$.
Hence, find the domain and range for $y = f(x^3)$.

$y = f(x^3) = x^3 + 1$. ✓
Domain: $\Re$ ✓
Range: $\Re$ ✓

(b) Express in terms of x, $y = f((x-1)^2)$.
Hence, find the domain and range for $y = f((x-1)^2)$.

$y = f((x-1)^2) = (x-1)^2 + 1$. ✓
Domain: $\Re$ ✓
Range: $\{y: y \ge 1, y \in \Re\}$ ✓

(c) Express in terms of x, $y = f(\sqrt{x+2})$.
Hence, find the domain and range for $y = f(\sqrt{x+2})$.

$y = f(\sqrt{x+2}) = \sqrt{x+2} + 1$. ✓
Domain: $\{x: x \ge -2, x \in \Re\}$ ✓
Range: $\{y: y \ge 1, y \in \Re\}$ ✓

6. [4 marks]

Consider the function $f(x) = \sqrt{4-x}$.
Express in terms of x, $y = f(2^x)$. Hence, find the domain and range for $y = f(2^x)$.

$y = f(2^x) = \sqrt{4-2^x}$. ✓
Clearly $4 - 2^x \ge 0$.
$\Rightarrow x \le 2$
Hence, domain: $\{x: x \le 2, x \in \Re\}$ ✓

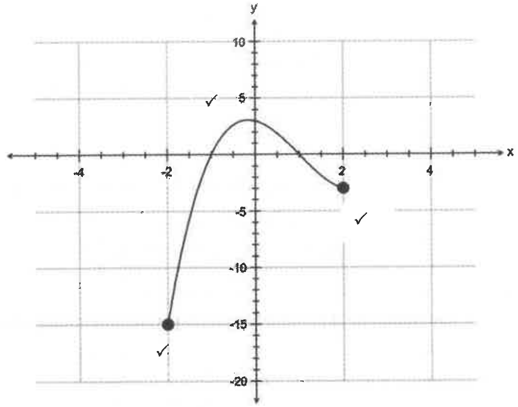
When $x = 2$, $y = 0$
As $x \to \infty$, $2^x \to 0$ and $y \to 2$.
Hence, range: $\{y: 0 \le y < 2, y \in \Re\}$ ✓✓

Calculator Assumed

7. [9 marks: 3, 2, 2, 2]

Consider the function $f(x) = x^3 - 3x^2 - x + 3$ for $-2 \le x \le 2$.

(a) In the axes provided below, sketch the graph of $y = f(x)$ within the specified domain.



(b) State the range for $f(x)$ for the domain specified. Give your answer correct to one decimal place.

Range: $-15.0 \le y \le 3.1$ ✓✓

(c) State the coordinates of the horizontal intercept(s) of $y = f(x)$ for the domain specified.

(–1, 0) and (1, 0) ✓✓
[–1 if (3, 0) mentioned.]

(d) State the coordinates of the turning point(s) of $y = f(x)$ for the domain specified. State the nature of this point. Give your answer correct to one decimal place.

(–0.2, 3.1) Max point ✓✓
[–1 if (2.2, –3.1) mentioned.]

10 Transformations on Curves

Calculator Free

1. [10 marks: 2, 2, 2, 2, 2]

Describe a sequence of transformations required to convert $y = f(x)$ into $y = g(x)$.

(a) $f(x) = x^2$ and $g(x) = (x-2)^2 + 4$

> Translate Right 2 units then translate Up 4 units. ✓✓

(b) $f(x) = x^3$ and $g(x) = -(2x)^3$

> Dilate parallel to positive x-axis factor ½ then reflect about the x-axis. ✓✓

(c) $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{1-x}$

> Reflect about the y-axis then translate right 1. ✓✓

(d) $f(x) = 3^x$ and $g(x) = -3^{x+1}$

> Translate 1 unit left, then reflect about the x-axis. ✓✓

(e) $f(x) = (2x+1)^2$ and $g(x) = x^2$.

> Dilate parallel to positive x-axis factor 2 then translate right 1 unit. ✓✓

2. [4 marks: 2, 2]

Describe a sequence of transformations required to transform:

(a) $x^2 + y^2 = 100$ into $(x+5)^2 + (y-6)^2 = 100$

> Translate Left 5 units then translate Up 6 units. ✓✓

(b) $(x-2)^2 + (y-1)^2 = 64$ into $(x+7)^2 + (y+3)^2 = 64$

> Translate Left 9 units then translate Down 4 units. ✓✓

Calculator Free

3. [4 marks: 2, 2]

The curve $y = 2^{x+1}$ is transformed into $y = g(x)$.

(a) State the sequence of transformations involved if $g(x) = 2^{0.5x-1}$.

> Translate Right 2 units. ✓
> Dilate along the x-axis by a factor of 2. ✓

(b) State the sequence of transformations involved if $g(x) = 3(2^x)$.

> In either order:
> Translate Right 1 unit. ✓
> Dilate along the y-axis by a factor of 3. ✓

4. [4 marks: 2, 2]

The curve $y = 1 + \frac{1}{x-2}$ is transformed into $y = g(x)$.

(a) State the sequence of transformations involved if $g(x) = \frac{2}{x-2}$.

> Translate Down 1 unit. ✓
> Dilate along the y-axis by a factor of 2. ✓

(b) State the sequence of transformations involved if $g(x) = -1 + \frac{1}{x+2}$.

> In either order:
> Reflect about the y-axis. ✓
> Reflect about the x-axis. ✓

Calculator Free

5. [10 marks: 2, 2, 2, 2, 2]

Identify the sequence of transformations required to map:

(a) $y = f(x)$ to $y = 2f(2x)$

> In either order:
> Dilate parallel to x-axis factor ½ ✓
> Dilate parallel to y-axis factor 2 ✓

(b) $y = f(x)$ to $y = f(2x + 1)$

> Translate left 1 unit ✓
> Dilate parallel to x-axis factor ½ ✓
> Or equivalent.

(c) $y = f(x)$ to $y = f(2(x + 1))$

> Rewrite as $y = f(2x + 2)$
> Translate left 2 unit ✓
> Dilate parallel to x-axis factor ½ ✓
> Or equivalent.

(d) $y = f(x)$ to $y = f(1 - x)$

> Translate left 1 unit ✓
> Reflect about the y-axis ✓

(e) $y = f(x)$ to $y = 1 - f(x)$

> Reflect about the x-axis ✓
> Translate up 1 unit ✓

Calculator Free

6. [6 marks: 2; 2, 2]

A parabola has equation $y = x^2 + 2x - 3$. Find the equation of the resulting curve:

(a) if the parabola is dilated by a factor of 2 along the x-axis.

> $y = \left(\frac{x}{2}\right)^2 + 2\left(\frac{x}{2}\right) - 3$ ✓✓

(b) if the parabola is reflected about the x-axis and then translated 2 units along the negative y-axis.

> $y = -(x^2 + 2x - 3) - 2$ ✓✓

(c) if the parabola is translated 1 unit along the positive x-axis and then reflected about the y-axis.

> $y = (-x - 1)^2 + 2(-x - 1) - 3$ ✓✓

---

7. [6 marks: 2, 2, 2]

The curve $y = 5^x$ is mapped to $y = g(x)$ by the following sequence of transformations. Find $g(x)$.

(a) a translation in the direction of the positive x-axis by 3 units followed by a translation in the direction of the positive y-axis by 2 units

> $y = 5^{x-3} + 2$ ✓✓

(b) a dilation in the direction of the positive x-axis by a factor of 2 followed by a translation in the direction of the positive x-axis by −2 units

> $y = 5^{\frac{x}{2}+1}$ ✓✓

(c) a reflection about the y-axis followed by a dilation in the direction of the positive x-axis by a factor of ½.

> $y = 5^{-2x}$ ✓✓

Calculator Free

8. [10 marks: 2, 2, 2, 2, 2]

A curve with equation $y = \sqrt{x}$ is transformed into $y = k\sqrt{(ax+b)} + c$ by the following sequences of transformations. State the values of k, a, b and c.

(a) A translation 5 units in the direction of the positive x-axis followed by a dilation parallel to the positive x-axis of factor 2.

$b = -5$	$k = 1$	
$a = \frac{1}{2}$	$c = 0$	✓✓ $-\frac{1}{2}$ per error round down

(b) **A dilation parallel to the positive x-axis of factor 2 followed by a translation 5 units in the direction of the positive x-axis.**

$a = \frac{1}{2}$	$k = 1$	
$b = \frac{-5}{2}$	$c = 0$	✓✓ $-\frac{1}{2}$ per error round down

(c) A translation 5 units in the direction of the negative y-axis followed by a reflection about the x-axis.

$c = 5$	$a = 1$	
$k = -1$	$b = 0$	✓✓ $-\frac{1}{2}$ per error round down

(d) A reflection about the x-axis followed by a translation 5 units in the direction of the negative y-axis

$c = -5$	$a = 1$	
$k = -1$	$b = 0$	✓✓ $-\frac{1}{2}$ per error round down

(e) A reflection about the y-axis followed by a dilation of factor 3 parallel to the positive y-axis.

$a = -1$	$b = 0$	
$k = 3$	$c = 0$	✓✓ $-\frac{1}{2}$ per error round down

Calculator Free

9. [4 marks: 2, 2]

The circle with equation $(x + 6)^2 + (y - 7)^2 = 81$ is transformed into the circle with equation $(x - a)^2 + (y - b)^2 = r^2$ by the following sequences of transformations. State the values of a, b and r.

(a) A translation 3 units in the direction of the positive x-axis followed by a translation 5 units in the direction of the negative y-axis.

$a = 3$	$b = 2$	$r = 9$ ✓✓ $-\frac{1}{2}$ per error round down

(b) A dilation of factor 2 parallel to the x-axis followed by a dilation of factor 2 parallel to the y-axis.

$a = 12$	$b = 14$	$r = 18$ ✓✓ $-\frac{1}{2}$ per error round down

10. [4 marks: 2, 2]

The parabola with equation $y^2 = x$ is transformed into the parabola with equation $y^2 = k(x - a)$ by the following sequences of transformations. State the values of a and k.

(a) A reflection about the y-axis followed by a reflection about the x-axis.

$a = 0$	$k = -1$	✓✓

(b) A translation 4 units in the direction of the positive x-axis followed by a reflection about the y-axis.

$a = -4$	$k = -1$	✓✓

Calculator Free

11. [14 marks: 3, 3, 4, 4]

The curve $y = f(x)$ has a minimum turning point at (–2, –1) and a maximum turning point at (4, 6). Find the minimum and maximum turning points of the following curves. In each case, explain clearly how you obtained your answer.

(a) $y = f(2x)$

> $y = f(2x)$ is obtained from $y = f(x)$ by dilating $y = f(x)$ along the x-axis by a factor of $\frac{1}{2}$. ✓
> Hence, minimum turning point is now (–1, –1) ✓
> and maximum turning point is now (2, 6). ✓

(b) $y = 2f(x)$

> $y = 2f(x)$ is obtained from $y = f(x)$ by dilating $y = f(x)$ along the y-axis by a factor of 2. ✓
> Hence, minimum turning point is now (–2, –2) ✓
> and maximum turning point is now (4, 12). ✓

(c) $y = 1 - f(x)$

> $y = 1 - f(x)$ is obtained from $y = f(x)$ by reflecting $y = f(x)$ about the x-axis and then translating the resulting curve 1 unit up. ✓✓
> Hence, minimum turning point is now (4, –5) ✓
> and maximum turning point is now (–2, 2). ✓

(d) $y = f(1 - x)$.

> $y = f(1 - x)$ is obtained from $y = f(x)$ by translating $y = f(x)$ 1 unit left and then reflecting the resulting curve about the y-axis. ✓✓
> Hence, minimum turning point is now (3, –1) ✓
> and maximum turning point is now (–3, 6). ✓

Calculator Assumed

12. [8 marks: 2, 2, 2, 2]

The curve $y = f(x)$ has a maximum point at (1, 5), a minimum point at (–5, 2) and intercepts at (0, 4) and (5, 0). The curve has no other turning points and intercepts.

(a) State the coordinates of the horizontal intercept(s) of the curve $y = f(-x - 1)$.

> Transformations are right one, then reflect about the y-axis.
> Hence, (–6, 0). ✓✓

(b) State the coordinates of a horizontal intercept of the curve $y = f(x + 1) - 2$.

> Transformations are left one, then down 2.
> Hence, (–6, 0). ✓✓

(c) State the coordinates of the vertical intercept(s) of the curve $y = 2f(x + 1)$.

> Transformations are left one, then dilate along the y-axis factor 2.
> Hence, (0, 10). ✓✓

(d) State the coordinates of the maximum and minimum point of $y = -f(-x)$.

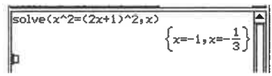
> Reflect about the y-axis, then reflect about the x-axis.
> Hence, maximum point is (5, –2) ✓
> and minimum point is (–1, –5). ✓

---

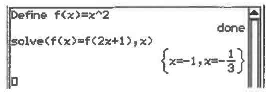
13. [3 marks]

Given that $f(x) = x^2$, solve $f(x) = f(2x + 1)$. Describe clearly how you obtained your answer.

> $f(x) = x^2$
> $\Rightarrow f(2x + 1) = (2x + 1)^2$.
> Hence, $f(x) = f(2x + 1)$ becomes $x^2 = (2x + 1)^2$ ✓
> $\Rightarrow x = -1/3, -1$ ✓✓



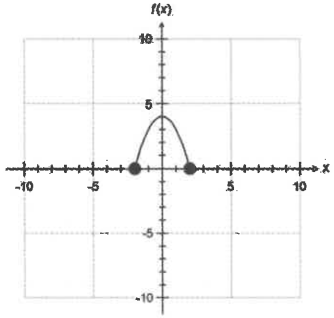
> OR
> Use "Define" command in CAS calculator.
> $x = -1/3, -1$ ✓✓✓



Calculator Assumed

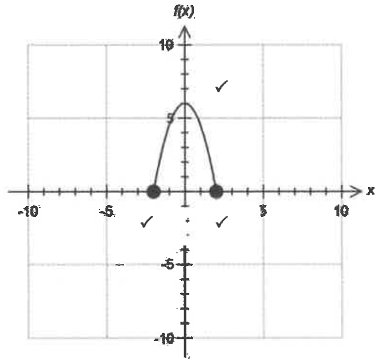
14. [6 marks: 3, 3]

The sketch of $y = f(x)$ is given in the accompanying diagram.

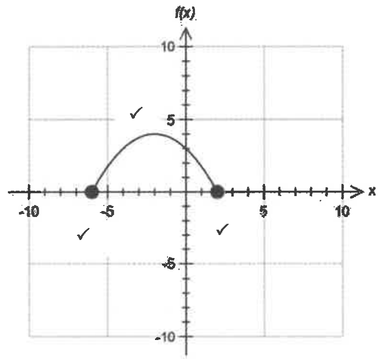


Sketch:

(a) $y = \frac{3}{2}f(x)$



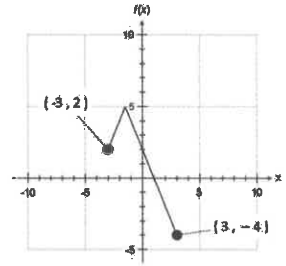
(b) $y = f(\frac{x}{2}+1)$



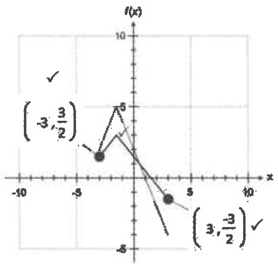
Calculator Assumed

15. [6 marks: 3, 3]

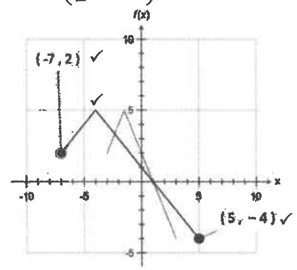
Given the graph of $y = f(x)$, sketch in the axes provided $y = g(x)$.



(a) $g(x) = \frac{1}{2}(f(x)+1)$

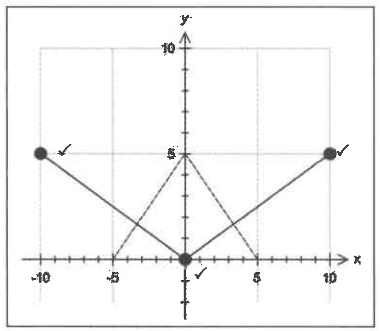


(b) $g(x) = f\left(\frac{1}{2}(x+1)\right)$



16. [3 marks]

Given the graph of $y = 5 - f(2x)$, sketch in the axes provided $y = f(x)$.



11 Equations

Calculator Free

1. [17 marks: 2, 2, 3, 3, 3, 4]

Solve for x:

(a) $2x - 5 = -3x + 4$

$5x = 9 \Rightarrow x = 9/5$ ✓✓

(b) $(2x - 5)(4 - 3x) = 0$

$x = 5/2, 4/3$ ✓✓

(c) $4x^2 - 49 = 0$

$x^2 = \frac{49}{4}$ ✓

$\Rightarrow x = \pm\frac{7}{2}$ ✓✓

(d) $x^2 + 1 = 4x - 3$

$x^2 - 4x + 4 = 0$ ✓

$(x - 2)^2 = 0$ ✓

$x = 2$ ✓

(e) $(2x - 1)^2 - 25 = 0$

$(2x - 1)^2 = 25$

$2x - 1 = \pm 5$ ✓

$x = -2$ or 3 ✓✓

(f) $x^2 + 4x - 3 = 0$

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2}$ ✓

$= -2 \pm \frac{\sqrt{28}}{2}$ ✓

$= -2 \pm \frac{2\sqrt{7}}{2}$ ✓

$= -2 \pm \sqrt{7}$ ✓

Calculator Free

2. [20 marks: 2, 2, 1, 3, 3, 5, 4]

Solve for real values of x:

(a) $(x - 5)(x + 3)(1 - 4x) = 0$

$x = 5, -3, \frac{1}{4}$ ✓✓

(b) $(x + 3)(x^2 - 36) = 0$

$x = -3, \pm 6$ ✓✓

(c) $(x^2 + 1)(2x - 5) = 0$

$x = 5/2$ ✓

(d) $(x^2 - 5x + 6)(3 - 2x) = 0$

$(x - 3)(x - 2)(3 - 2x) = 0$ ✓

$x = 3, 2, 3/2$ ✓✓

(e) $x^3 = x^2 + 2x$

$x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$ ✓

$x(x - 2)(x + 1) = 0$

$x = 0, 2, -1$ ✓✓

(f) $x^3 + 4x^2 - 7x - 10 = 0$

For $x = 1$, $x^3 + 4x^2 - 7x - 10 = -12$

$x = -1$, $x^3 + 4x^2 - 7x - 10 = 0$. ✓

Hence $x^3 + 4x^2 - 7x - 10 = (x + 1)(x^2 + 3x - 10)$ ✓✓

$= (x + 1)(x + 5)(x - 2)$

Hence, roots are: $x = -5, -1, 2$ ✓✓

(g) $2x^3 + 5x^2 - 4x - 3 = 0$

For $x = 1$, $2x^3 + 5x^2 - 4x - 3 = 0$

Hence $2x^3 + 5x^2 - 4x - 3 = (x - 1)(2x^2 + 7x + 3)$ ✓✓

$= (x - 1)(2x + 1)(x + 3)$

Hence, roots are: $x = -3, -\frac{1}{2}, 1$ ✓✓

Calculator Free

3. [15 marks: 3, 3, 3, 3, 3]

Solve for x:

(a) $\frac{3}{x} = x + 2$

$3 = x^2 + 2x$
$x^2 + 2x - 3 = 0$ ✓
$(x+3)(x-1) = 0 \quad \Rightarrow x = -3, 1$ ✓✓

(b) $\frac{2}{x-1} = \frac{1}{x+4}$

$2(x+4) = x - 1$ ✓
$2x + 8 = x - 1$ ✓
$\Rightarrow x = -9$ ✓

(c) $\frac{-1}{x+1} = x + 3$

$-1 = (x+3)(x+1)$
$x^2 + 4x + 4 = 0$ ✓
$(x+2)^2 = 0 \quad \Rightarrow x = -2$ ✓✓

(d) $\frac{1}{x} = x + 1$

$1 = x(x+1)$
$x^2 + x - 1 = 0$ ✓
$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$ ✓
$x = \frac{-1 \pm \sqrt{5}}{2}$ ✓

(e) $x - 5 = \frac{1}{x-1}$

$(x-5)(x-1) = 1$
$x^2 - 6x + 4 = 0$ ✓
$x = \frac{6 \pm \sqrt{6^2 - 4(1)(4)}}{2}$ ✓
$x = 3 \pm \frac{\sqrt{20}}{2}$ ✓
$= 3 \pm \sqrt{5}$

Calculator Free

4. [12 marks: 2, 2, 2, 3, 3]

Solve for real values of x:

(a) $\sqrt{x+1} = 5$

$x + 1 = 25$ ✓
$x = 24$ ✓

(b) $\sqrt{x^2+16} = 5$

$x^2 + 16 = 25$ ✓
$x = \pm 3$ ✓✓

(c) $\sqrt[3]{2x+3} = 2$

$2x + 3 = 8$ ✓
$x = 5/2$ ✓

(d) $\sqrt{5-4x} = x$

$5 - 4x = x^2$
$x^2 + 4x - 5 = 0$ ✓
$(x-1)(x+5) = 0$
$x = 1$ ✓
Reject -5 as $\sqrt{25} \neq -5$ ✓

(e) $x = \sqrt{4x-3}$

$x^2 = 4x - 3$
$x^2 - 4x + 3 = 0$ ✓
$(x-3)(x-1) = 0$
$x = 1, 3$ ✓✓

5. [9 marks: 2, 2, 2, 3]

Solve simultaneously for x and y (where possible):

(a) $x + y = 10, x = -4$

$x = -4, y = 14$ ✓✓

(b) $x + y = 10, x - y = 8$

$x + y = 10$ (1)
$x - y = 8$ (2)
(1) + (2) $2x = 18$
$x = 9, y = 1$ ✓✓

Calculator Free

5. (c) $2x + y = 10,\ 4x + 2y = 8$

$$\begin{array}{lll} & 2x + y = 10 & (1) \\ & 4x + 2y = 8 & (2) \\ (1)\times 2 & 4x + 2y = 20 & (1b) \\ (1b) - (2) & 0 = 12 \text{ (impossible!)} & \checkmark \\ \text{Hence, no solution.} & & \checkmark \end{array}$$

(d) $2x + 3y = 4,\ 3x + y = -1$

$$\begin{array}{llll} & 2x + 3y = 4 & (1) & \\ & 3x + y = -1 & (2) & \\ (2)\times 3 & 9x + 3y = -3 & (2b) & \checkmark \\ (2b) - (1) & 7x = -7 & & \\ & x = -1,\ y = 2 & & \checkmark\checkmark \end{array}$$

---

6. [10 marks: 2, 2, 2, 2, 2]

Solve simultaneously for x and y where x and y are both integers:

(a) $x^2 + y^2 = 10,\ x = -1$

$x = -1,\ y = \pm 3$ ✓✓

(b) $(x-1)^2 + (y+2)^2 = 13,\ y = 1$

$$\begin{array}{ll} (x-1)^2 + 9 = 13 & \\ (x-1)^2 = 4 \Rightarrow x = -1, 3 & \\ \text{Hence, } x = -1,\ y = 1 & \checkmark \\ x = 3,\ y = 1 & \checkmark \end{array}$$

(c) $x^2 + y^2 = 2,\ x + y = 0$

By inspection: $x = 1,\ y = -1$ ✓
$x = -1,\ y = 1$ ✓

(d) $x^2 + y^2 = 5,\ x + y = 3$

By inspection: $x = 1,\ y = 2$ ✓
$x = 2,\ y = 1$ ✓

(e) $x^2 + y^2 = 41,\ x + y = 9$

By inspection: $x = 5,\ y = 4$ ✓
$x = 4,\ y = 5$ ✓

Calculator Assumed

7. [9 marks: 1, 1, 2, 2, 3]

Solve for x in exact form.

(a) $\dfrac{5x}{3} + \dfrac{1}{7} = \dfrac{3}{4} - \dfrac{2x}{5}$

$x = \dfrac{255}{868}$ ✓

(b) $\dfrac{(x+2)}{4} + 3 = \dfrac{1}{3} - \dfrac{5(x-7)}{2}$

$x = \dfrac{172}{33}$ ✓

(c) $2x^2 - 15 = 0$

$x = \pm\dfrac{\sqrt{30}}{2}$ ✓✓

(d) $x^2 + 2x - 5 = 0$

$x = -1 \pm \sqrt{6}$ ✓✓

(e) $x^3 + 2x^2 - 11x - 12 = 0$

$x = -4, -1, 3$ ✓✓✓

---

8. [8 marks: 1, 1, 1, 2, 3]

Solve for x correct to 4 decimal places.

(a) $\dfrac{(2x-1)}{7} - x = \dfrac{2x}{3} - \dfrac{3(x+5)}{4}$

$x = 5.7170$ ✓

(b) $\sqrt[3]{7x-2} = 5$

$x = 18.1429$ ✓

(c) $\sqrt{3+2x} = x$

$x = 3$ ✓

Calculator Assumed

8. (d) $(x+5)^2 - 11 = 0$

$x = -8.3166, -1.6834$ ✓✓

(e) $x^3 + 3x^2 = 4x + 10$

$x = -3.2924, -1.6027, 1.8951$ ✓✓✓

---

9. [14 marks: 2 each]

Solve for x and y to the specified accuracy:

(a) $y = \frac{1}{x}$ and $y = x - 3$ (Two decimal places)

$x = -0.30, y = -3.30$ and $x = 3.30, y = 0.30$ ✓✓

(b) $y = -x + 5$ and $y = \frac{5}{2x-1}$ (Two decimal places)

$x = 1.15, y = 3.85$ and $x = 4.35, y = 0.65$ ✓✓

(c) $5x + 3y = 10, 7x - 8y = -12$ (Exact Answers)

$x = 44/61, y = 130/61$ ✓✓

(d) $1.2x - 3.5y = -0.9, 6.1x + 3.6y = 4.2$ (Four decimal places)

$x = 0.4464, y = 0.4102$ ✓✓

(e) $x^2 + y^2 = 1, x + y = 0$ (Exact Answers)

$x = \pm\frac{\sqrt{2}}{2}, y = \mp\frac{\sqrt{2}}{2}$ ✓✓

(f) $x^2 + y^2 = 5, x - y = 2$ (Two decimal places)

$x = -0.22, y = -2.22$ ✓
$x = 2.22, y = 0.22$ ✓

(g) $(x-1)^2 + (y+1)^2 = 2, x + y = 1$ (Two decimal places)

$x = 0.63, y = 0.37$ ✓
$x = 2.37, y = -1.37$ ✓

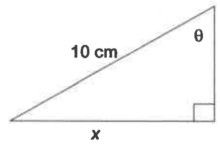
12 Right Triangle Trigonometry

Calculator Free

1. [4 marks: 2, 2]

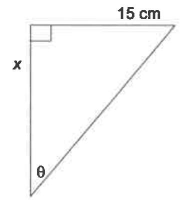
Given that $\sin\theta = 0.6$, $\cos\theta = 0.8$, $\tan\theta = 0.75$, find x in the following triangles:

(a)



$\sin\theta = \frac{x}{10}$ ✓
$x = 10 \times \sin\theta$
$= 10 \times 0.6 = 6$ cm ✓

(b)

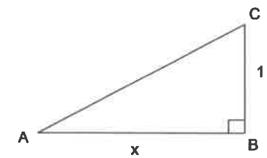


$\tan\theta = \frac{15}{x}$ ✓
$x = \frac{15}{\tan\theta}$
$= \frac{15}{0.75} = 20$ cm ✓

---

2. [5 marks: 1, 1, 3]

For ΔABC as shown, find :



(a) AC in terms of x.

$AC = \sqrt{(x^2+1)}$ ✓

(b) $\cos \angle CAB$ in terms of x.

$\cos \angle CAB = \frac{x}{\sqrt{x^2+1}}$ ✓

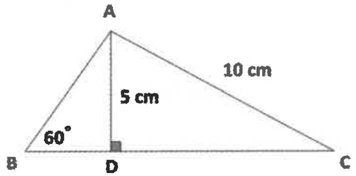
(d) the exact value of x if $\cos \angle CAB = \frac{\sqrt{7}}{7}$.

$\frac{x}{\sqrt{x^2+1}} = \frac{\sqrt{7}}{7} \Rightarrow 7x = \sqrt{(7x^2+7)}$ ✓
$49x^2 = 7x^2 + 7$
$x = \frac{\sqrt{6}}{6}$ (reject $-\frac{\sqrt{6}}{6}$) ✓✓

Calculator Free

3. [7 marks: 2, 2, 3]

In ΔABC, $\angle ABD = 60^\circ$, $AD = 5$ cm and $AC = 10$ cm. AD is perpendicular to BC.



(a) Find BD.

$\tan 60^\circ = \dfrac{5}{BD}$ ✓

$\Rightarrow BD = \dfrac{5}{\tan 60^\circ} = \dfrac{5}{\sqrt{3}} = \dfrac{5\sqrt{3}}{3}$ ✓

(b) Find $\angle DAC$.

$\cos \angle DAC = \dfrac{5}{10}$ ✓

$\Rightarrow \angle DAC = \cos^{-1}\dfrac{1}{2} = 60^\circ$ ✓

(c) Find BC.

$DC = 10 \sin \angle DAC = 10 \sin 60^\circ$ ✓

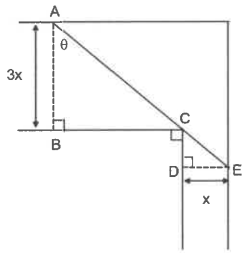
$= 10 \times \dfrac{\sqrt{3}}{2} = 5\sqrt{3}$ ✓

$BC = BD + DC = \dfrac{5\sqrt{3}}{3} + 5\sqrt{3} = \dfrac{20\sqrt{3}}{3}$ ✓

4. [4 marks]

Triangles ABC and CDE are right-angled triangles with BC parallel to DE and AB parallel to CD. $DE = x$ and $AB = 3x$.

Prove that $AE = x\left[\dfrac{1}{\sin\theta} + \dfrac{3}{\cos\theta}\right]$.



In ΔABC, $AC = \dfrac{3x}{\cos\theta}$. ✓

In ΔCDE, $CE = \dfrac{x}{\sin\theta}$ ✓

$AE = AC + CE \Rightarrow AE = \dfrac{3x}{\cos\theta} + \dfrac{x}{\sin\theta}$ ✓

$= x\left[\dfrac{3}{\cos\theta} + \dfrac{1}{\sin\theta}\right]$. ✓

Calculator Assumed

5. [9 marks: 1, 2, 6]

A light aircraft flies horizontally at a speed of 120 kmh^{-1}. During the flight, the pilot noted that it took the plane 30 seconds to fly from being at an angle of depression of 40° to a farmhouse to being directly overhead.

(a) Find the horizontal distance between the aircraft and the farmhouse at the instant the angle of depression to the farmhouse is 40°.

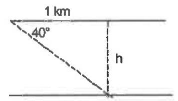
Horizontal distance $= 120 \times \dfrac{30}{60 \times 60}$

$= 1$ km ✓

(b) Find the altitude of the aircraft.

$\dfrac{h}{1} = \tan 40$ ✓

$\Rightarrow h = 0.8391 = 0.84$ km ✓



(c) Immediately after passing the farmhouse, the aircraft climbs at an angle of 15° to the horizon for 2 minutes. Find the angle of elevation of the aircraft from the farmhouse at the end of the two minutes.

Distance travelled $= 120 \times \dfrac{2}{60}$

$= 4$ km ✓

In ΔABC, vertical rise $r = 4 \sin 15$

$= 1.0353$ km ✓

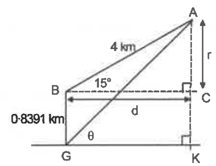
In ΔABC, horizontal increase $d = 4 \cos 15$

$= 3.8637$ km ✓

In ΔAGK, $AK = 1.0353 + 0.8391$

$= 1.8744$ km ✓

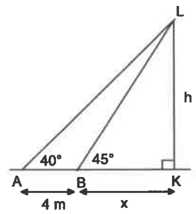
Hence, $\tan\theta = \dfrac{1.8744}{3.8637} \Rightarrow \theta = 25.9^\circ$ ✓✓



Calculator Assumed

6. [4 marks]

A ball is caught between the branches of a tree. The angle of elevation of the ball from a point A on the ground is 40°. From a second point B on the ground, 4 metres closer to the foot of the tree than A, the angle of elevation of the ball is 45°. Assume that A, B and the ball are in the same vertical plane. Find the vertical distance between the ball and the ground.



In $\triangle LAK$, $h = (4 + x)\tan 40$ I ✓

In $\triangle LBK$, $h = x \tan 45 = x$ ✓

Hence, from I, $h = (4 + h)\tan 40$ ✓

$h = 4\tan 40 + h\tan 40$

$h(1 - \tan 40) = 4\tan 40$

$$h = \frac{4\tan 40}{1 - \tan 40}$$

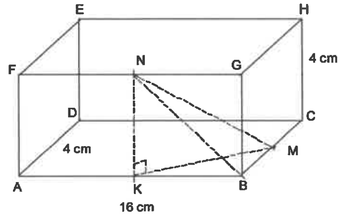
$h = 20.9$ m ✓

```
solve(x=(4+x)tan(40),x)
                {x=20.86010461}
```


Calculator Assumed

7. [10 marks: 2, 2, 2, 4]

In the rectangular box shown, M and N are the midpoints of BC and FG respectively. AB = 16 cm, AD = 4 cm and HC = 4 cm. Let K be the midpoint of AB. Find:



(a) the exact length of MK.

In $\triangle KBM$, $MK = \sqrt{(8^2 + 2^2)}$
$= \sqrt{68}$
$= 2\sqrt{17}$ ✓✓

(b) the exact length of MN.

In $\triangle NKM$, $MN^2 = NK^2 + KM^2$ ✓
$= 4^2 + 68$
Hence, $MN = \sqrt{84} = 2\sqrt{21}$ ✓

(c) the angle between MN and the plane ABCD.

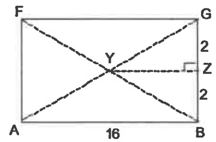
Required angle is $\angle NMK$.
In $\triangle NKM$, $\sin\angle NMK = \frac{NK}{NM} = \frac{4}{2\sqrt{21}}$ ✓
Hence, $\angle NMK = 25.9°$ ✓

(d) the acute angle between the planes EFBC and ADHG.

Let the diagonals BF and AG meet at Y.
Required angle is $\angle GYB$. ✓

In $\triangle YGZ$, $\tan\angle GYZ = \frac{2}{8}$. ✓

$\Rightarrow \angle GYZ = 14.04°$ ✓
Hence, $\angle GYB = 2 \times \angle GYZ$
$= 2 \times 14.04 = 28.08 = 28.1°$ ✓



13 Non-Right Triangle Trigonometry

Calculator Free

1. [4 marks: 2, 2]

Given that $\sin 45^\circ = \dfrac{\sqrt{2}}{2}$ and $\sin 60^\circ = \dfrac{\sqrt{3}}{2}$:

(a) find x if $\dfrac{x}{\sin 60^\circ} = \dfrac{15}{\sin 45^\circ}$.

$x = \dfrac{15\sin 60}{\sin 45} = \dfrac{15\sqrt{6}}{2}$ ✓✓

(b) find $\sin\theta$ if $\dfrac{10}{\sin\theta} = \dfrac{15}{\sin 45^\circ}$.

$\sin\theta = \dfrac{10\sqrt{2}}{2} \times \dfrac{1}{15} = \dfrac{\sqrt{2}}{3}$ ✓✓

2. [8 marks: 2, 2, 3]

(a) Find $x > 0$ if $x^2 = (\sqrt{2})^2 + 3^2 - 2 \times \sqrt{2} \times 3 \times \cos\theta$ where $\cos\theta = \dfrac{1}{\sqrt{2}}$.

$x^2 = 2 + 9 - 6\sqrt{2} \times \dfrac{1}{\sqrt{2}}$ ✓
$x = \sqrt{5}$ ✓

(b) Find $\cos\theta$ in exact form if $8^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta$.

$\cos\theta = \dfrac{16+25-64}{40} = \dfrac{-23}{40}$ ✓✓

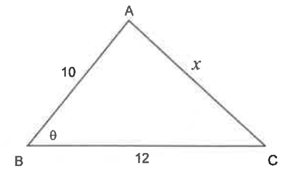
(c) Find $x > 0$ if $(\sqrt{3})^2 = 1^2 + x^2 - 2x\cos 60^\circ$.

$3 = 1 + x^2 - 2x(1/2)$ ✓
$x^2 - x - 2 = 0$
$(x+1)(x-2) = 0$ ✓
$x = 2$ (reject $x = -1$) ✓

Calculator Free

3. [6 marks: 2, 2, 2]

In triangle ABC drawn below, find:



(a) the exact value of x if $\cos\theta = \dfrac{1}{2}$.

$x^2 = 10^2 + 12^2 - 2(10)(12)\cos\theta$ ✓
$= 124$
$x = \sqrt{124} = 2\sqrt{31}$ ✓

(b) $\cos\theta$ in exact form if $x = 12$.

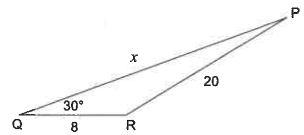
$\cos\theta = \dfrac{10^2+12^2-12^2}{2(10)(12)} = \dfrac{5}{12}$ ✓✓

(c) Find the area of ΔABC if $\sin\theta = \dfrac{\sqrt{2}}{2}$.

Area = ½ × 10 × 12 × sin θ ✓
$= 30\sqrt{2}$ ✓

4. [5 marks: 2, 3]

In the accompanying ΔPQR:



(a) find the exact value of $\sin\angle$QPR.

$\dfrac{\sin Q\hat{P}R}{8} = \dfrac{\sin 30}{20}$ ✓
$\sin\angle QPR = \dfrac{8}{20} \times \dfrac{1}{2} = \dfrac{1}{5}$ ✓

(b) show that the length of the side PQ satisfies the equation $x^2 - 8\sqrt{3}\,x - 336 = 0$.

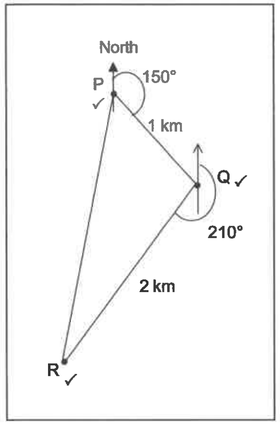
$20^2 = 8^2 + x^2 - 2(8)(x)\cos 30$ ✓
$400 = 64 + x^2 - \dfrac{16\sqrt{3}}{2}x$ ✓
$x^2 - (8\sqrt{3})x - 336 = 0$ ✓

Calculator Free

5. [7 marks: 3, 1, 3]

P, Q and R are three spots on a large level farm land. Q is located 1 km from P along bearing 150°. R is located 2 km from Q along bearing 210°.

(a) Draw a clearly labelled diagram indicating relative positions of P, Q and R. State all relevant angles and distances.



(b) Find the bearing of P from Q.

N30°W ✓

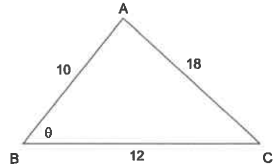
(c) Find in exact form the distance between P and R.

$PR^2 = 1^2 + 2^2 - 2(1)(2)\cos 120$ ✓✓
$= 1 + 4 + 2$
$PR = \sqrt{7}$ ✓

Calculator Assumed

6. [3 marks: 2, 1]

In triangle ABC shown, find:
(a) the exact value of $\cos\theta$.



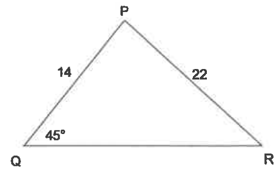
$\cos\theta = \dfrac{10^2 + 12^2 - 18^2}{2(10)(12)}$
$= -\dfrac{80}{240} = -\dfrac{1}{3}$ ✓✓

(b) θ giving your answer(s) to the nearest 0.1 of a degree.

$\theta = 109.5^\circ$ ✓

7. [3 marks: 2, 1]

In triangle PQR shown, find:
(a) the exact value of $\sin\angle PRQ$.



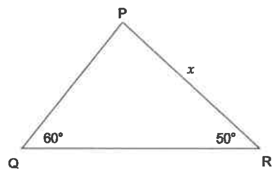
$\dfrac{\sin P\hat{R}Q}{14} = \dfrac{\sin 45}{22}$ ✓
$\sin\angle PRQ = \dfrac{7\sqrt{2}}{22}$ ✓

(b) $\angle PRQ$ giving your answer(s) to the nearest 0.1 of a degree.

$\angle PRQ = 26.7^\circ$ ✓

8. [4 marks: 2, 2]

In triangle PQR shown, find:
(a) the length of PQ in terms of x.



$\dfrac{PQ}{\sin 50} = \dfrac{x}{\sin 60}$ ✓
$PQ = \dfrac{x \sin 50}{\sin 60} = 0.8846x$ ✓

(b) the length of QR in terms of x.

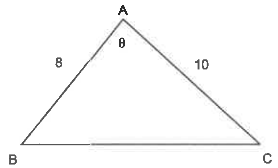
$\dfrac{QR}{\sin 70} = \dfrac{x}{\sin 60}$ ✓
$QR = \dfrac{x \sin 70}{\sin 60} = 1.0851x$ ✓

Calculator Assumed

9. [4 marks: 2, 2]

In triangle ABC shown, find:

(a) the length of BC in terms of θ.



$BC^2 = 8^2 + 10^2 - 2(8)(10)\cos\theta$ ✓
$= 164 - 160\cos\theta$
$BC = \sqrt{(164 - 160\cos\theta)}$
$= 2\sqrt{(41 - 40\cos\theta)}$ ✓

(b) the size of ∠ACB if θ = 80°.

$BC = 2\sqrt{(41 - 40\cos 80)}$
$\dfrac{\sin A\hat{C}B}{8} = \dfrac{\sin 80}{2\sqrt{41 - 40\cos 80}}$ ✓
$\sin\angle ACB = 0.6750$
$\angle ACB = 42.5^\circ$ ✓

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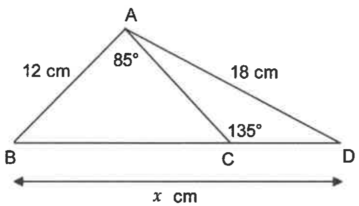
10. [11 marks: 3, 3, 3, 2]

In the accompanying diagram:

(a) find BC to 4 decimal places.



$\angle ACB = 180^\circ - 135^\circ = 45^\circ$ ✓
$\dfrac{BC}{\sin 85} = \dfrac{12}{\sin 45}$ ✓
$BC = 16.9060$ ✓

(b) find AC to 4 decimal places.

$\angle ABC = 180^\circ - 85^\circ - 45^\circ = 50^\circ$ ✓
$\dfrac{AC}{\sin 50} = \dfrac{12}{\sin 45}$ ✓
$AC = 13.0002$ ✓

(c) find CD to 3 decimal places.

In △ACD, let CD = y.
$18^2 = 13.0002^2 + y^2 - 2(13.0002)(y)\cos 135$ ✓✓
$y = 6.283$ (reject −24.668) ✓

(d) hence, find x to 2 decimal places.

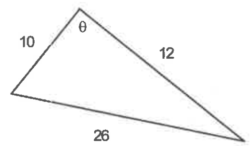
Hence, $x = 16.906 + 6.283 = 23.189 = 23.19$ cm ✓✓

Calculator Assumed

11. [4 marks]

Use the cosine rule to prove that it is impossible to have a triangle with sides measuring 10 cm, 12 cm and 26 cm.

Let θ be the largest angle in the triangle.
Then, θ will be the angle opposite the longest side of the triangle.



$\cos\theta = \dfrac{10^2 + 12^2 - 26^2}{2(10)(12)}$ ✓
$= -1.8$ ✓
But $-1 \le \cos\theta \le 1$. ✓
Hence, it is not possible to have a triangle with such measurements. ✓

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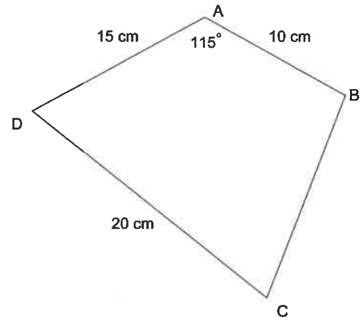
12. [5 marks]

Find the area of quadrilateral ABCD given that ∠BDC = 40°.



$BD^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \times \cos 115$ ✓
$BD = 21.2552$ ✓

Area of ABCD = Area of △ABD + Area of △BDC
$= \dfrac{1}{2} \times 15 \times 10 \times \sin 115 + \dfrac{1}{2} \times 20 \times 21.2552 \times \sin 40$ ✓✓
$= 204.5989 = 204.6\ \text{cm}^2$. ✓

Calculator Assumed

13. [5 marks]

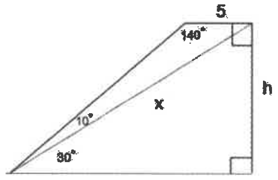
An aeroplane flying at a constant altitude (height) of h km is sighted at an angle of elevation of 40°. A few minutes later the plane had a flown a further 5 km and is sighted at an angle of elevation 30°. Find h.

Diagram ✓

$\frac{x}{\sin 140} = \frac{5}{\sin 10}$ ✓

$x = 18.5083$ ✓

$h = x \sin 30$ ✓
$= 9.2542$
$= 9.3$ km ✓



14. [9 marks: 6, 3]

From where James is, the referee is 10 metres away on bearing 50°. From where Chris is, the referee is on bearing 290°. James is 30 metres from Chris.

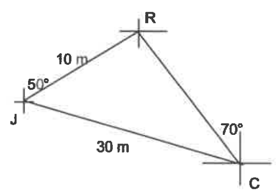
(a) Find the distance between Chris and the referee.

Diagram ✓✓

In $\triangle JRC$, $\angle JRC = 50^\circ + 70^\circ = 120^\circ$ ✓

Let $RC = y$.
$30^2 = 10^2 + y^2 - 2(10)(y)\cos 120$ ✓
$y^2 + 10y - 800 = 0$ ✓

Hence, $y = 23.7228$ (reject -33.7228)
$= 23.7$ metres ✓



(b) Find the bearing of Chris from James.

$\frac{\sin R\hat{J}C}{23.7228} = \frac{\sin 120}{30}$ ✓

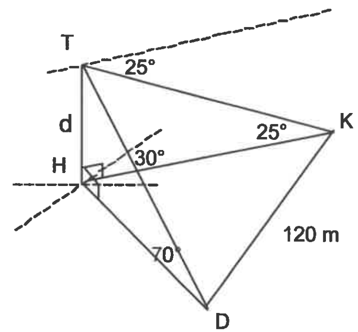
$\angle RJC = 43.22^\circ$ (reject 136.78°) ✓

Hence, bearing is $50^\circ + 43.22^\circ = 093.2^\circ$ ✓

Calculator Assumed

15. [5 marks]

From the top of an observation tower of height d metres, a kangaroo is spotted on the ground on an angle of depression of 25° along bearing 030°. A dingo is also spotted on the ground on an angle of depression of 70° along bearing 110°. The dingo is estimated to be 120 metres away from the kangaroo.



Find the height of the observation tower.

In $\triangle THK$, $HK = \frac{d}{\tan 25}$. ✓

In $\triangle THD$, $HD = \frac{d}{\tan 70}$. ✓

In $\triangle HKD$,
$\angle KHD = 110^\circ - 30^\circ = 80^\circ$

$KD^2 = HD^2 + HK^2 - 2(HD)(HK)\cos 80$

$120^2 = \left(\frac{d}{\tan 25}\right)^2 + \left(\frac{d}{\tan 70}\right)^2 - 2\left(\frac{d}{\tan 70}\right)\left(\frac{d}{\tan 25}\right)\cos 80$ ✓✓

$d = 56.82$ metres (reject -56.82) ✓

14 Arcs, Sectors & Segments

Calculator Free

1. [4 marks]

Complete the following table.

Angle in Degrees	Angle in Radians (exact values)
0	0
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
120	$2\pi/3$
150	$5\pi/6$
180	π

✓✓✓✓

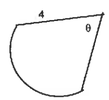
2. [4 marks: 2, 2]

A circular sector is removed from a circle of radius 4 cm.

(a) Find the angle of this circular sector (in radians) if the area of this sector is 2π cm^2.

$\frac{1}{2} \times (4)^2 \times \theta = 2\pi$ ✓

$\theta = \pi/4$ ✓



(b) Find the angle of this circular sector (in radians) if the perimeter of the sector is 16 cm.

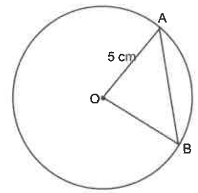
$4\theta + 8 = 16$ ✓

$\theta = 2$ radians ✓

Calculator Free

3. [8 marks: 2, 2, 4]

In the circle of radius 5 cm with centre O drawn below, $\angle AOB = 60^\circ$.



(a) Find the *exact* area of triangle OAB

Area of $\triangle ABC = \frac{1}{2} \times 5 \times 5 \times \sin 60$

$= \frac{25\sqrt{3}}{4}$ ✓✓

(b) Find the *exact* area of the minor segment formed by the chord AB.

Area of minor segment $= \frac{1}{2} \times 5^2 \times \frac{\pi}{3} - \frac{25\sqrt{3}}{4}$

$= \frac{25\pi}{6} - \frac{25\sqrt{3}}{4}$ ✓✓

(c) Find the exact perimeter of the minor segment formed by the chord AB.

Δ OAB is equilateral as $\angle AOB = 60^\circ$.
Hence, AB = 5 cm. ✓✓

Length of minor arc AB $= 5 \times \frac{\pi}{3} = \frac{5\pi}{3}$ cm. ✓

Hence, perimeter $= 5 + \frac{5\pi}{3}$ cm. ✓

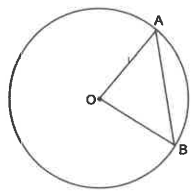
Calculator Assumed

4. [8 marks: 2, 3 3]

In the circle of radius 2π cm with centre O

$\angle AOB = \frac{\pi}{3}$.

Find the *exact* (as a multiple or fraction of π) :



(a) area of triangle OAB

$$\text{Area} = \frac{1}{2} \times 2\pi \times 2\pi \times \sin\frac{\pi}{3} \quad ✓$$
$$= \pi^2\sqrt{3} \quad ✓$$

(b) perimeter of the *major* segment formed by the chord AB.

$$\text{Perimeter} = 2\pi + 2\pi \times \frac{5\pi}{3} \quad ✓✓$$
$$= 2\pi + \frac{10\pi^2}{3} \quad ✓$$

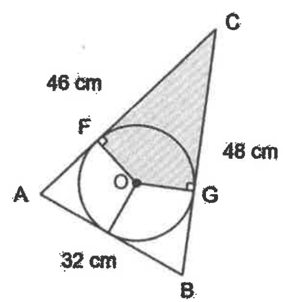
(c) area of the major segment formed by the chord AB.

$$\text{Area} = \frac{1}{2}\times(2\pi)^2\times\frac{5\pi}{3} + \pi^2\sqrt{3} \quad ✓✓$$
$$= \frac{10\pi^3}{3} + \pi^2\sqrt{3} \quad ✓$$

Calculator Assumed

5. [11 marks: 3, 2, 2, 4]

The accompanying diagram shows a circle of radius 11 cm enclosed within triangle ABC. The circle touches all three sides of the triangle.



(a) Find the size of $\angle ACB$.
Give your answer to the nearest degree.

Using the cosine rule:
$$\cos\angle ACB = \frac{46^2+48^2-32^2}{2(46)(48)} \quad ✓$$
$$\angle ACB = 39.73 \quad ✓$$
$$\approx 40° \quad ✓$$

(b) Hence, find the obtuse $\angle FOG$. Give your answer to the nearest degree.

$$\text{Obtuse } \angle FOG = 180 - 39.73 \quad ✓$$
$$= 140.27$$
$$\approx 140° \quad ✓$$

(c) Find the area of the minor sector FOG.

$$\text{Area} = \frac{140.27}{360}\times\pi\times 11^2 \quad ✓$$
$$= 148.11 \text{ cm}^2 \quad ✓$$

(d) Find the area of the shaded region.
Show clearly how you obtained your answer.

In ΔOGC:
$$CG = \frac{11}{\tan\left(\frac{39.73}{2}\right)} \quad ✓$$
$$= 30.4453 \quad ✓$$

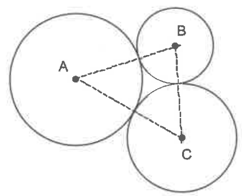
Hence, area of $\Delta OGC = \frac{1}{2}\times 11\times 30.4453$ ✓

Area of shaded region $= 2\times\frac{1}{2}\times 11\times 30.4453$
$$= 334.9 \text{ cm}^2 \quad ✓$$

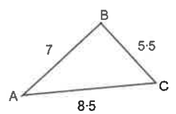
Calculator Assumed

6. [10 marks: 5, 5]

Three circles of radii 5 cm, 3.5 cm and 2 cm are drawn touching each other as shown in the accompanying diagram.



(a) Find the size of all angles within triangle ABC.



$\cos\angle BAC = \dfrac{7^2 + 8.5^2 - 5.5^2}{2(7)(8.5)}$ ✓

$\Rightarrow \angle BAC = 40.12^\circ$ ✓

$\cos\angle ABC = \dfrac{7^2 + 5.5^2 - 8.5^2}{2(7)(5.5)}$ ✓

$\Rightarrow \angle ABC = 84.78^\circ$ ✓

$\angle ACB = 180^\circ - 40.12^\circ - 84.78^\circ = 55.10^\circ$ ✓

(b) Find the area of the region trapped by the three circles.

Area of $\triangle ABC = \frac{1}{2} \times 7 \times 8.5 \times \sin 40.12$
$= 19.17 \text{ cm}^2$ ✓

Area of sector with centre at A $= \dfrac{40.12}{360} \times \pi \times 5^2$
$= 8.75 \text{ cm}^2$ ✓

Area of sector with centre at B $= \dfrac{84.78}{360} \times \pi \times 2^2$
$= 2.96 \text{ cm}^2$ ✓

Area of sector with centre at C $= \dfrac{55.10}{360} \times \pi \times 3.5^2$
$= 5.89 \text{ cm}^2$ ✓

Required Area $= 19.17 - (8.75 + 2.96 + 5.89)$
$= 1.57 \text{ cm}^2$ ✓

15 Trigonometric Equations I

Calculator Free

1. [0 marks]

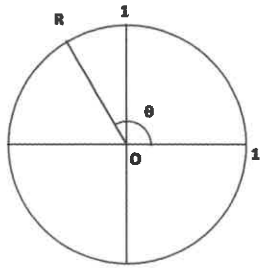
Complete the following table. Give answers in exact form.

Angle θ in degrees	Angle θ in radians	sin θ	cos θ	tan θ
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	∞
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
180°	π	0	-1	0
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270°	$\frac{3\pi}{2}$	-1	0	$-\infty$
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
360°	2π	0	1	0

Calculator Free

2. [9 marks: 1, 3, 1, 2, 2]

The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates $(-\frac{1}{3}, k)$.



(a) Find cos θ.

$\cos\theta = x\text{-coordinate of R}$
$= -\frac{1}{3}$ ✓

(b) Find the two possible values of k.

R is on the unit circle.
Hence $\left(-\frac{1}{3}\right)^2 + k^2 = 1$ ✓
$\frac{1}{9} + k^2 = 1$
$k = \pm\frac{2\sqrt{2}}{3}$ ✓✓

(b) Hence, find:
(i) sin θ.

$\sin\theta = y\text{-coordinate of R}$
$= \pm\frac{2\sqrt{2}}{3}$ ✓

(ii) tan θ.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$= \frac{\pm\frac{2\sqrt{2}}{3}}{-\frac{1}{3}}$ ✓
$= \pm 2\sqrt{2}$ ✓

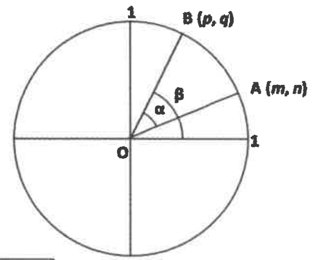
(iii) sin (180° – θ).

If θ is in quadrant 2, then 180° – θ is in quadrant 1.
⇒ $\sin(180^\circ - \theta) = \frac{2\sqrt{2}}{3}$ ✓
If θ is in quadrant 3, then 180° – θ is a negative angle in quadrant 4.
⇒ $\sin(180^\circ - \theta) = -\frac{2\sqrt{2}}{3}$ ✓

Calculator Free

3. [10 marks: 1, 1, 2, 3, 3]

The angle β is defined by the ray OB. The angle α is the angle trapped between the rays OA and OB. O is the centre of the unit circle. A is a point on the unit circle with coordinates (m, n). B is a point on the unit circle with coordinates (p, q). m, n, p and q are all positive numbers.



(a) Find cos (180° – β).

$\cos(180^\circ - \beta) = -\cos\beta$
$= -p$ ✓

(b) Find sin (–β).

$\sin -\beta = -\sin\beta$
$= -q$ ✓

(c) Find sin (90° – β).

$\sin(90^\circ - \beta) = \cos\beta$ ✓
$= p$ ✓

(d) Find cos (90° + β).

90° + β is in Q2.
Reference angle for $90^\circ + \beta = 180^\circ - (90^\circ + \beta)$
$= 90^\circ - \beta$
$\cos(90^\circ + \beta) = -\cos(90^\circ - \beta)$ ✓
$= -\sin\beta$ ✓
$= -q$ ✓

(e) Find tan (β – α).

$\sin(\beta - \alpha) = n$ ✓
$\cos(\beta - \alpha) = m$ ✓
Hence, $\tan(\beta - \alpha) = \frac{n}{m}$ ✓

Calculator Free

4. [5 marks: 2, 3]

Find the equation of the line:

(a) passing through the origin and inclined at an angle of 30° with the positive x-axis.

> Equation is $y = x \tan 30^\circ$ ✓
> $= \frac{\sqrt{3}}{3}x$ ✓

(b) passing through the point $(\sqrt{3}, 4)$ and inclined at an angle of 60° with the positive x-axis.

> Equation is $(y-4) = \tan 60^\circ (x - \sqrt{3})$ ✓✓
> $y = x\sqrt{3} + 1$ ✓

---

5. [6 marks: 3, 3]

Given that $\tan 15^\circ = -\sqrt{3} + 2$, find the equation of the line:

(a) passing through the origin, inclined at an angle of 165° with the positive x-axis.

> Equation is $y = x \tan 165^\circ$ ✓
> $= x(-\tan 15^\circ)$ ✓
> $= -x(-\sqrt{3} + 2)$
> $= (\sqrt{3} - 2)x$ ✓

(b) passing through the origin, inclined at an angle of 75° with the positive x-axis.

> Equation is $y = x \tan(90^\circ - 15^\circ)$ ✓
> $= x(\tan 75^\circ)$
> $= \left(\frac{1}{-\sqrt{3}+2}\right)x$ ✓
> $= (\sqrt{3} + 2)x$ ✓

Calculator Free

6. [18 marks: 3, 3, 3, 5, 4]

Solve for θ within the given domain:

(a) $\sin\theta = \frac{\sqrt{3}}{2}$ where $0^\circ \le \theta \le 360^\circ$

> Reference angle for $\theta = 60^\circ$. ✓
> θ is in Quadrant 1 and Quadrant 2.
> Hence, $\theta = 60^\circ, 180^\circ - 60^\circ$
> $= 60^\circ, 120^\circ$. ✓✓

(b) $\cos\theta = -\frac{\sqrt{2}}{2}$ where $0 \le \theta \le 2\pi$

> Reference angle for $\theta = \frac{\pi}{4}$. ✓
> θ is in Quadrant 2 and Quadrant 3.
> Hence, $\theta = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$
> $= \frac{3\pi}{4}, \frac{5\pi}{4}$ ✓✓

(c) $\tan\theta = \sqrt{3}$ where $-\pi < \theta \le \pi$

> Reference angle for $\theta = \frac{\pi}{3}$. ✓
> θ is in Quadrant 1 and Quadrant 3.
> Hence, $\theta = \frac{\pi}{3}, -\pi + \frac{\pi}{3}$
> $= \frac{\pi}{3}, -\frac{2\pi}{3}$ ✓✓

(d) $\sin(2\theta) = -0.5$ where $-\pi \le \theta \le \pi$.

> Reference angle for $2\theta = \pi/6$. ✓
> 2θ is in Quadrant 3 and Quadrant 4.
> Hence, $2\theta = -\pi/6, -5\pi/6, -2\pi - \pi/6, -2\pi - 5\pi/6$ ✓✓
> $\theta = -\pi/12, -5\pi/12, -13\pi/12, -17\pi/12$ ✓
> $= -\pi/12, -5\pi/12, 11\pi/12, 7\pi/12$ ✓

(e) $\sin\theta = \cos\theta$ where $-180^\circ < \theta \le 180^\circ$

> $\sin\theta = \cos\theta \Rightarrow \tan\theta = 1$ ✓
> Reference angle for $\theta = 45^\circ$. ✓
> θ is in Quadrant 1 and Quadrant 3.
> Hence, $\theta = 45^\circ, -180^\circ + 45^\circ$
> $= 45^\circ, -135^\circ$. ✓✓

Calculator Free

7. [17 marks: 4, 4, 4, 5]

(a) Given that $\cos 66.4^\circ = 0.4$, solve for θ in $\cos(\theta + 30^\circ) = 0.4$ where $0^\circ \le \theta \le 360^\circ$.

> Reference angle for $\theta + 30^\circ = 66.4^\circ$. ✓
>
> $\theta + 30^\circ$ is in Quadrant 1 and Quadrant 4.
>
> Hence, $\theta + 30^\circ = 66.4^\circ, 360^\circ - 66.4^\circ$ ✓
>
> $\theta = 36.4^\circ, 263.6^\circ$. ✓✓

(b) Given that $\tan 26.6^\circ = 0.5$, solve for θ in $1 - 2\tan(\theta + 6.6^\circ) = 0$ where $0 \le \theta \le 360^\circ$.

> Rewrite as $2\tan(\theta + 6.6^\circ) = 1$
>
> $\Rightarrow \tan(\theta + 6.6^\circ) = ½$ ✓
>
> Reference angle for $\theta + 6.6^\circ = 26.6^\circ$. ✓
>
> $\theta + 6.6^\circ$ is in Quadrant 1 and Quadrant 3.
>
> Hence, $\theta + 6.6^\circ = 26.6^\circ, 180^\circ + 26.6^\circ$ ✓
>
> $\theta = 20^\circ, 200^\circ$. ✓

(c) $(\sin\theta - 2)(2\sin\theta - 1) = 0$ where $0^\circ \le \theta \le 360^\circ$

> $\Rightarrow \sin\theta = 2$ or $\sin\theta = ½$ ✓✓
>
> $\sin\theta = 2$ gives no solution.
>
> Hence, $\sin\theta = ½ \Rightarrow \theta = 30^\circ, 150^\circ$ ✓✓

(d) $2\cos^2\theta + 3\cos\theta - 2 = 0$ where $0 \le \theta \le 2\pi$

> Factorise, $(2\cos\theta - 1)(\cos\theta + 2) = 0$ ✓
>
> $\cos\theta = \frac{1}{2}$ or $\cos\theta = -2$ ✓✓
>
> $\cos\theta = -2$ gives no solution.
>
> Hence, $\cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3, 5\pi/3$ ✓✓

Calculator Assumed

8. [4 marks: 2, 2]

(a) Find the angle the line with equation $y = 2x + 5$ makes with the positive x-axis.

> Gradient of line $m = 2$.
>
> Let the required angle be θ.
>
> Hence, $\tan\theta = 2$ ✓
>
> $\theta = 63.43 \approx 63.4^\circ$ ✓

(b) Line L has equation $3x + 4y = 12$.
Find the angle this line makes with the positive x-axis.

> Gradient of line $m = -0.75$.
>
> Let the required angle be θ.
>
> Hence, $\tan\theta = -0.75$ ✓
>
> $\theta = 143.13 \approx 143.1^\circ$ ✓

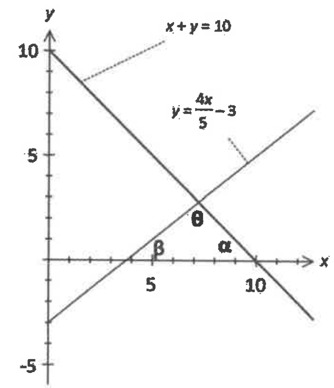
---

9. [5 marks]

Lines L1 and L2 have equations $x + y = 10$ and $y = \frac{4x}{5} - 3$ respectively.

Find the acute angle between these two lines.

> Gradient of L1, $m = -1$.
>
> Hence, acute angle L1 makes with the positive x-axis, $\alpha = 45^\circ$. ✓
>
> Gradient of L2, $m = \frac{4}{5}$.
>
> Hence, acute angle L2 makes with the positive x-axis, $\beta = \tan^{-1}\frac{4}{5}$ ✓
>
> $\approx 38.66^\circ$. ✓
>
> Therefore, angle between lines,
>
> $\theta \approx 180^\circ - \alpha - \beta$
>
> $\approx 96.34^\circ$ ✓
>
> Acute angle between lines $\approx 83.66^\circ$ ✓



16 Trigonometric Graphs

Calculator Free

1. [6 marks]

Complete the following table.

Function	Period	Amplitude (where applicable)
$y = 2\sin(2x^{\circ})$	180°	2
$y = -4\cos(\frac{x}{2} + 30^{\circ})$	720°	4
$v = 10\tan(3t + \pi)$	$\frac{\pi}{3}$	n/a
$Q = 5\sin(\frac{\pi}{2} - t)$	2π	5
$y = \frac{\sqrt{2}}{2}\cos(\pi t) + 100$	2	$\frac{\sqrt{2}}{2}$
$T = 5 - \sin(\frac{\pi}{4} - \theta)$	2π	1

[–1/2 per error, round down]

2. [5 marks]

Complete the table below.

Function	Minimum value of function	Maximum value of function
$y = 3\sin t$	–3	3
$y = 20\cos(\frac{2x}{3} - 45^{\circ})$	–20	20
$v = 5\tan\theta$	n/a	n/a
$M = 2\sin(\frac{\pi}{2} - 3t) + 4$	$-2 + 4 = 2$	$2 + 4 = 6$
$y = 5 - \cos(2\pi t)$	$5 - 1 = 4$	$5 - (-1) = 6$

[–1/2 per error, round down]

Calculator Free

3. [8 marks: 4, 4]

A trigonometric function has equation $y = -4\sin(2x + 30^{\circ})$ for $0^{\circ} \le x \le 360^{\circ}$. Find:

(a) the maximum value for y and the corresponding value(s) for x.

> Maximum value for $y = -4 \times -1 = 4$. ✓
>
> This occurs when $\sin(2x + 30^{\circ}) = -1$ ✓
>
> $\Rightarrow 2x + 30^{\circ} = 270^{\circ}, 630^{\circ}$
>
> $x = 120^{\circ}, 300^{\circ}$ ✓✓

(b) the minimum value for y and the corresponding values for x.

> Minimum value for $y = -4 \times 1 = -4$. ✓
>
> This occurs when $\sin(2x + 30^{\circ}) = 1$ ✓
>
> $\Rightarrow 2x + 30^{\circ} = 90^{\circ}, 450^{\circ}$
>
> $x = 30^{\circ}, 210^{\circ}$ ✓✓

4. [4 marks]

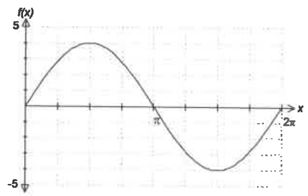
A trigonometric function has equation $P = a\cos(bt + \frac{\pi}{4})$. Find the values of a and b given that P has a maximum value of 4 and a period of 4.

> $a = \pm 4$ ✓✓
>
> Period $\frac{2\pi}{b} = 4$ ✓
>
> $b = \frac{\pi}{2}$ ✓

Calculator Assumed

5. [3 marks]

The accompanying diagram shows the graph of a trigonometric function. State the amplitude and period of the function. Hence, give the equation of this function.

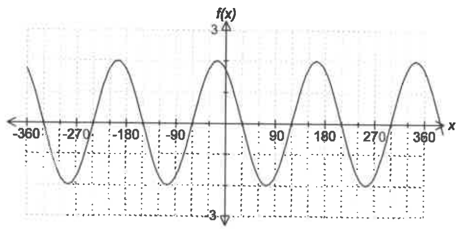


Amplitude = 4 ✓
Period = 2π ✓
$f(x) = 4\sin x$ ✓

6. [7 marks: 4, 3]

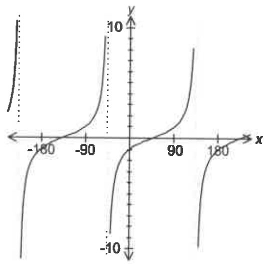
Find the equation of the following trigonometric functions:

(a)



Equation is $f(x) = 2\cos(2x + 30^\circ)$ ✓✓✓✓

(b)



Equation is $y = \tan(x - 45^\circ)$ ✓✓✓

Calculator Assumed

7. [11 marks: 1, 1, 2, 2, 5]

The body temperature θ (Celsius) of a reptile in summer at time *t* hours after midnight is given by $\theta = 15 - 5\sin(\frac{\pi t}{12})$.

(a) State the period for θ.

Period = $\frac{2\pi}{\pi/12}$ = 24 hours ✓

(b) What is the range of body temperature experienced by the reptile?

10° Celsius to 20° Celsius. Range = 10°. ✓

(c) Find the minimum body temperature of the reptile and state when this first occurs after midnight.

Minimum body temperature is 10° Celsius. ✓

This occurs when $\sin(\frac{\pi t}{12}) = 1$.

$\Rightarrow \frac{\pi t}{12} = \frac{\pi}{2} \Rightarrow t = 6$ ✓

Hence, the minimum of 10° Celsius first occurs at 6.00 am.

(d) Find the maximum body temperature of the reptile and state when this first occurs after midnight.

Maximum temperature is 20 Celsius. ✓

This occurs when $\sin(\frac{\pi t}{12}) = -1$.

$\Rightarrow \frac{\pi t}{12} = \frac{3\pi}{2} \Rightarrow t = 18$ ✓

Hence, the maximum of 20° Celsius first occurs at 6.00 pm

(e) Use an algebraic method to find the first time when the temperature of the reptile is 16° Celsius.

$15 - 5\sin(\frac{\pi t}{12}) = 16 \Rightarrow \sin(\frac{\pi t}{12}) = -\frac{1}{5}$ ✓✓

Reference angle for $\frac{\pi t}{12} = 0.2014$

$\frac{\pi t}{12} = \pi + 0.2014$ (Quadrant 3) ✓

$t = 12.7691$ = 12 hours 46 minutes ✓

Hence, this first occurs at 12.46 pm. ✓

17 Trigonometric Identities (Add/Sub Formulae)

Calculator Free

1. [13 marks: 4, 4, 5]

Use an appropriate trigonometric identity to find the exact value of :

(a) sin 75°

$$\sin 75 = \sin(30+45) = \sin 30\cos 45 + \cos 30 \sin 45 \quad ✓$$
$$= \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} \quad ✓✓$$
$$= \frac{\sqrt{2}(1+\sqrt{3})}{4} \quad ✓$$

(b) cos 165°

$$\cos 165 = \cos(120+45) = \cos 120\cos 45 - \sin 120 \sin 45 \quad ✓$$
$$= -\frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} \quad ✓✓$$
$$= -\frac{\sqrt{2}(1+\sqrt{3})}{4} \quad ✓$$

(c) $\tan\frac{7\pi}{12}$

$$\tan\frac{7\pi}{12} = \tan\left[\frac{\pi}{3}+\frac{\pi}{4}\right]$$
$$= \frac{\tan\frac{\pi}{3}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{3}\tan\frac{\pi}{4}} \quad ✓$$
$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{\sqrt{3}+1}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}} \quad ✓✓$$
$$= \frac{(\sqrt{3}+1)^2}{-2} \quad ✓$$
$$= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3} \quad ✓$$

Calculator Free

2. [10 marks: 2, 1, 3, 4]

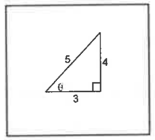
Given that $\sin A = \frac{4}{5}$ and $0 < A < \frac{\pi}{2}$, find the exact value of:

(a) cos A

A is an acute angle.

From the triangle sketched, with $\sin A = \frac{4}{5}$,

$\cos A = \frac{3}{5}$ ✓✓



(b) tan A

From the triangle sketched in part (a),

$\tan A = \frac{4}{3}$ ✓

(c) $\sin(\frac{\pi}{2}+A)$

$$\sin(\frac{\pi}{2}+A) = \sin\frac{\pi}{2}\cos A + \cos\frac{\pi}{2}\sin A \quad ✓$$
$$= \cos A \quad ✓$$
$$= \frac{3}{5} \quad ✓$$

(d) $\cos(\frac{\pi}{4}-A)$

$$\cos(\frac{\pi}{4}-A) = \cos\frac{\pi}{4}\cos A + \sin\frac{\pi}{4}\sin A \quad ✓$$
$$= \frac{\sqrt{2}}{2}\times\frac{3}{5} + \frac{\sqrt{2}}{2}\times\frac{4}{5} \quad ✓✓$$
$$= \frac{\sqrt{2}}{2}(\frac{3}{5}+\frac{4}{5})$$
$$= \frac{7\sqrt{2}}{10} \quad ✓$$

Calculator Free

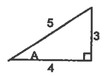
3. [13 marks: 2, 2, 3, 3, 3]

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{4}$, where A and B are acute, find the exact value of:

(a) cos A

A is an acute angle. From the triangle sketched,

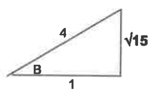
with $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$ ✓✓



(b) sin B

B is an acute angle. From the triangle sketched,

with $\cos B = \frac{1}{4}$, $\sin B = \frac{\sqrt{15}}{4}$ ✓✓



(c) sin(A + B)

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \frac{3}{5} \times \frac{1}{4} + \frac{4}{5} \times \frac{\sqrt{15}}{4}$ ✓✓

$= \frac{3 + 4\sqrt{15}}{20}$ ✓

(d) cos (A – B)

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= \frac{4}{5} \times \frac{1}{4} + \frac{3}{5} \times \frac{\sqrt{15}}{4}$ ✓✓

$= \frac{4 + 3\sqrt{15}}{20}$ ✓

(e) tan (A + B)

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{3}{4} + \sqrt{15}}{1 - \frac{3}{4}\sqrt{15}}$ ✓✓

$= \frac{3 + 4\sqrt{15}}{4 - 3\sqrt{15}}$ ✓

Calculator Assumed

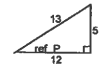
4. [13 marks: 2, 2, 3, 3, 3]

Given that $\sin P = \frac{5}{13}$ and $\cos Q = -\frac{15}{17}$, where P and Q are each obtuse angles, find the exact value of:

(a) cos P

P is an obtuse angle. From the triangle sketched,

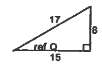
with sin (reference angle for P) $= \frac{5}{13}$, $\cos P = -\frac{12}{13}$ ✓✓



(b) sin Q

Q is an obtuse angle. From the triangle sketched,

with cos (reference angle for Q) $= \frac{15}{17}$, $\sin Q = \frac{8}{17}$ ✓✓



(c) sin(P – Q)

$\sin(P - Q) = \sin P \cos Q - \cos P \sin Q$

$= \frac{5}{13} \times (-\frac{15}{17}) - (-\frac{12}{13}) \times \frac{8}{17}$ ✓✓

$= \frac{21}{221}$ ✓

(d) cos (P + Q)

$\cos(P + Q) = \cos P \cos Q - \sin P \sin Q$

$= (-\frac{12}{13}) \times (-\frac{15}{17}) - \frac{5}{13} \times \frac{8}{17}$ ✓✓

$= \frac{140}{221}$ ✓

(e) tan (P – Q)

$\tan(P - Q) = \frac{\tan P - \tan Q}{1 + \tan P \tan Q}$

$= \frac{\left[-\frac{5}{12}\right] - \left[-\frac{8}{15}\right]}{1 + \left[-\frac{5}{12}\right] \times \left[-\frac{8}{15}\right]}$ ✓✓

$= \frac{\frac{7}{60}}{\frac{11}{9}} = \frac{21}{220}$ ✓

18 Trigonometric Equations II

Calculator Free

1. [13 marks: 3, 5, 5]

Solve for x within the given domain:

(a) $\cos x + \sqrt{3}\sin x = 0 \qquad 0 \le x \le 360^\circ$:

$\sqrt{3}\sin x = -\cos x$
$\tan x = -1/\sqrt{3}$ ✓

Reference angle for $x = 30^\circ$. ✓
$\Rightarrow x = 150^\circ, 330^\circ$ ✓

(b) $2\sin^2 x - 3\sin x - 2 = 0 \qquad$ for $0 \le x \le 360^\circ$

$2\sin^2 x - 3\sin x - 2 = 0$
$(2\sin x + 1)(\sin x - 2) = 0$ ✓
$\Rightarrow \sin x = -½$ or 2 (reject) ✓✓

$\sin x = -½$
Reference angle for $x = 30^\circ$.
Angle x is in Quadrant 3 or Quadrant 4.

Hence, $x = 180^\circ + 30^\circ, 360^\circ - 30^\circ$
$= 210^\circ, 330^\circ$ ✓✓

(c) $\cos x - \dfrac{3}{\cos x} - 2 = 0 \qquad$ for $0 \le x \le 2\pi$

Multiply both sides of equation with $\cos x$.
$\Rightarrow \cos^2 x - 2\cos x - 3 = 0$ ✓
$(\cos x + 1)(\cos x - 3) = 0$ ✓
$\Rightarrow \cos x = -1$ or 3 (reject) ✓✓

$\cos x = -1 \Rightarrow x = \pi$ radians ✓

Calculator Free

2. [16 marks: 5, 5, 6]

Solve for θ within the given domain:

(a) $\cos(\theta + 30^\circ) = \sin\theta \qquad$ for $0 \le \theta \le 360^\circ$

$\cos(\theta + 30^\circ) = \sin\theta$
$\cos\theta\cos 30^\circ - \sin\theta\sin 30^\circ = \sin\theta$ ✓
$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \sin\theta$
$\frac{3}{2}\sin\theta = \frac{\sqrt{3}}{2}\cos\theta$ ✓
$\tan\theta = \frac{\sqrt{3}}{3}$ ✓
$\theta = 30^\circ, 210^\circ$ ✓✓

OR
First value for θ may be obtained by inspection!

(b) $\sin(\theta + \frac{\pi}{4}) = \sqrt{2}\cos\theta \qquad$ for $0 \le \theta \le 2\pi$

$\sin(\theta + \frac{\pi}{4}) = \sqrt{2}\cos\theta$
$\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} = \sqrt{2}\cos\theta$ ✓
$\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta = \sqrt{2}\cos\theta$
$\frac{\sqrt{2}}{2}\sin\theta = \frac{\sqrt{2}}{2}\cos\theta$ ✓
$\tan\theta = 1$ ✓
$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ ✓✓

OR
First value for θ may be obtained by inspection!

(c) $\sin(\theta - \frac{\pi}{4}) = -\sqrt{2}\cos(\theta + \frac{\pi}{6}) \qquad$ for $0 \le \theta \le 2\pi$

$\sin(\theta - \frac{\pi}{4}) = \sqrt{2}\cos(\theta + \frac{\pi}{6})$
$\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4} = -\sqrt{2}(\cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6})$ ✓
$\frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta = -\sqrt{2}(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta)$ ✓
Divide each term by $\frac{\sqrt{2}}{2}$:
$\sin\theta - \cos\theta = -\sqrt{3}\cos\theta + \sin\theta$ ✓
$(\sqrt{3} - 1)\cos\theta = 0 \Rightarrow \cos\theta = 0$ ✓
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ✓✓

Calculator Free

3. [14 marks: 2, 5, 7]

(a) Use the formula for sin (A + B) to show that sin 2A = 2 sin A cos A.

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

Let A = B. ✓
Hence, $\sin(A + A) = \sin A \cos A + \cos A \sin A$ ✓
$\sin 2A = 2 \sin A \cos A$.

(b) Use the formula in (a) to solve for x in $\cos x + \sin 2x = 0$ for $0 \le x \le 360°$.

$\cos x + \sin 2x = 0$
$\cos x + 2 \sin x \cos x = 0$ ✓
$\cos x (1 + 2 \sin x) = 0$
$\Rightarrow \cos x = 0$ or $\sin x = -\frac{1}{2}$ ✓✓

$\cos x = 0 \Rightarrow x = 90°, 270°$ ✓

$\sin x = -\frac{1}{2} \Rightarrow x = 210°, 330°$ ✓

(c) Use the formula in (a) to solve for x in $\sin 2x - \sin x = 0$ for $0 < \theta < 2\pi$.

Rewrite as $2 \sin x \cos x - \sin x = 0$. ✓
$\sin x (2 \cos x - 1) = 0$ ✓
$\Rightarrow \sin x = 0$ or $\cos x = \frac{1}{2}$ ✓✓

$\sin x = 0 \Rightarrow x = \pi$ ✓

$\cos x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$ ✓✓

19 Sets

Calculator Assumed

1. [6 marks: 1, 1, 2, 2]

Given that **U** = { x | $50 \le x \le 70$, x is an integer},
A = {51, 53, 65, 68} , **B** = {62, 64, 65, 66} and **C** = {51, 53, 66, 70}.

(a) Find |**U**|.

|**U**| = 20 + 1 = 21 ✓

(b) Find **A** ∪ **B**.

A ∪ **B** = {51, 53, 62, 64, 65, 66, 68} ✓

(c) Find n(**C** ∩ $\overline{\mathbf{B}}$).

C ∩ $\overline{\mathbf{B}}$ = {51, 53, 70} ✓
Hence, n(**C** ∩ $\overline{\mathbf{B}}$) = 3 ✓

(d) |Find $\overline{\mathbf{A \cap B \cap C}}$ |.

A ∩ **B** ∩ **C** = ϕ ✓
Hence, | $\overline{\mathbf{A \cap B \cap C}}$ | = 21 − 0 = 21 ✓

2. [7 marks: 1, 2, 2, 2]

Given that **U** = { x | $0 \le x \le 20$, x is an integer} ,
P = { x | $5 \le x \le 14$} , **Q** = { x | $3 \le x \le 9$} and **R** = {2, 4, 6}.

(a) Is 2 ∈ **R**?

Yes! ✓

(b) Is **Q** ⊂ **P**? Justify your answer.

No! ✓ 3 which is an element of Q is not in P. ✓

(c) Find n(**Q**').

Q = {3, 4, 5, 6, 7, 8, 9} ⇒ n(**Q**) = 7 ✓
n($\overline{\mathbf{Q}}$) = n(**U**) − n(**Q**) = 21 − 7 = 14 ✓

(d) Find |**U** ∩ **P**|.

U ∩ **P** = **P** ✓
Hence, |**U** ∩ **P**| = |**P**| = 10 ✓

Calculator Assumed

3. [6 marks: 1, 1, 2, 2]

Given that $\mathbf{U} = \{ x \mid -10 \le x \le 10, x \text{ is an integer}\}$,
$\mathbf{A} = \{ x \mid -10 \le x \le -1\}$, $\mathbf{B} = \{ x \mid 0 \le x \le 10\}$ and $\mathbf{C} = \{2, 3, 5, 7\}$.

(a) Find $\mathbf{A} \cap \mathbf{B}$.

> $\mathbf{A} \cap \mathbf{B} = \phi$ ✓

(b) Find $\mathbf{A} \cup \mathbf{B}$.

> $\mathbf{A} \cup \mathbf{B} = \mathbf{U}$ ✓

(c) Find $n(\mathbf{B} \cap \overline{\mathbf{A}})$.

> $\overline{\mathbf{A}} = \mathbf{B} \Rightarrow \mathbf{B} \cap \overline{\mathbf{A}} = \mathbf{B}$ ✓
> Hence $n(\mathbf{B} \cap \overline{\mathbf{A}}) = n(\mathbf{B}) = 11$ ✓

(d) Find $|(\mathbf{A} \cup \mathbf{B}) \cap \overline{\mathbf{C}}|$.

> Since $\mathbf{A} \cup \mathbf{B} = \mathbf{U}$, $\mathbf{A} \cup \mathbf{B} \cap \overline{\mathbf{C}} = \overline{\mathbf{C}}$ ✓
> Hence $|\overline{\mathbf{C}}| = 21 - 4 = 17$ ✓

---

4. [8 marks: 2, 2, 2, 2]

Given that $\mathbf{U} = \{ x \mid 1 \le x \le 20, x \text{ is an integer}\}$, $\mathbf{A} = \{ x \mid x \text{ is a prime number} \}$,
$\mathbf{B} = \{ x \mid x \text{ is a square number}\}$ and $\mathbf{C} = \{ x \mid x \text{ is a multiple of } 3\}$.

(a) Find $\mathbf{B} \cap \mathbf{C}$.

> $\mathbf{B} = \{1, 4, 9, 16\}$ $\mathbf{C} = \{3, 6, 9, 12, 15, 18\}$ ✓
> Hence, $\mathbf{B} \cap \mathbf{C} = \{9\}$ ✓

(b) Find $\mathbf{A} \cup \mathbf{B}$.

> $\mathbf{A} = \{2, 3, 5, 7, 11, 13, 17, 19\}$ $\mathbf{B} = \{1, 4, 9, 16\}$ ✓
> Hence, $\mathbf{A} \cup \mathbf{B} = \{1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19\}$ ✓

(c) Find $|\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})|$.

> $\mathbf{A} = \{2, 3, 5, 7, 11, 13, 17, 19\}$ $\mathbf{B} \cap \mathbf{C} = \{9\}$
> Hence, $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = \{2, 3, 5, 7, 9, 11, 13, 17, 19\}$ ✓
> $\Rightarrow |\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})| = 9$ ✓

(d) Find $|(\mathbf{A} \cup \mathbf{B}) \cap \mathbf{C})|$.

> $\mathbf{A} \cup \mathbf{B} = \{1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19\}$, $\mathbf{C} = \{3, 6, 9, 12, 15, 18\}$ ✓
> Hence, $\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = \{3, 9\}$
> $\Rightarrow |\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})| = 2$ ✓

Calculator Assumed

5. [9 marks: 2, 2, 3, 2]

Given that $\mathbf{U} = \{ x \mid 0 \le x \le 20, x \text{ is an integer}\}$, $\mathbf{A} = \{x \mid x \text{ is a factor of } 24\}$,
$\mathbf{B} = \{x \mid x \text{ is a prime number}\}$ and $\mathbf{C} = \{x \mid x \text{ is a triangular number}\}$.

(a) Find $\mathbf{B} \cap \mathbf{C}$

> $\mathbf{B} = \{2, 3, 5, 7, 11, 13, 17, 19\}$
> $\mathbf{C} = \{1, 3, 6, 10, 15\}$ ✓
> Hence, $\mathbf{B} \cap \mathbf{C} = \{3\}$ ✓

(b) Find $\mathbf{C} \cup \mathbf{B}$

> $\mathbf{B} = \{2, 3, 5, 7, 11, 13, 17, 19\}$
> $\mathbf{C} = \{1, 3, 6, 10, 15\}$
> Hence, $\mathbf{C} \cup \mathbf{B} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 15, 17, 19\}$ ✓✓

(c) $n(\mathbf{A} \cap \overline{\mathbf{B}})$

> $\mathbf{A} = \{1, 2, 3, 4, 6, 8, 12\}$
> $\overline{\mathbf{B}} = \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$ ✓
> $\mathbf{A} \cap \overline{\mathbf{B}} = \{1, 4, 6, 8, 12\} \Rightarrow |\mathbf{A} \cap \overline{\mathbf{B}}| = 5$ ✓✓

(d) An element is chosen at random from **U**. Find the probability that this element is from set **B**, given that it is from set **C**.

> $P(\mathbf{B} \mid \mathbf{C}) = \frac{1}{5}$ ✓✓

---

6. [5 marks: 1, 1, 1, 2]

Given that $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{B} = \{0, 1, 2\}$ and $\mathbf{C} = \{ (x, y) \mid x \in A, y \in B \}$.

(a) Find $\mathbf{A} \cap \mathbf{B}$

> $\mathbf{A} \cap \mathbf{B} = \{1, 2\}$ ✓

(b) Find $\mathbf{A} \cup \mathbf{B}$

> $\mathbf{A} \cup \mathbf{B} = \{0, 1, 2, 3\}$ ✓

(c) Is $(1, 2) \in C$?

> Yes! ✓

(d) $|\mathbf{C}|$

> $\mathbf{C} = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$
> Hence, $|\mathbf{C}| = 9$ ✓✓

20 Combinations

Calculator Free

1. [10 marks: 1, 2, 2, 2, 3]

Evaluate each of the following:

(a) 5!

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ✓

(b) $\binom{10}{5}$

$\binom{10}{5} = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}$ ✓
$= 252$ ✓

(c) $\binom{10}{5} \times 5!$

$\binom{10}{5} \times 5! = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} \times 5!$ ✓
$= 30\ 240$ ✓

(d) $\binom{40}{2}$

$\binom{40}{2} = \frac{40\times39}{2\times1}$ ✓
$= 780$ ✓

(e) $\binom{60}{58}$

$\binom{60}{58} = \binom{60}{2}$ ✓
$= \frac{60\times59}{2\times1}$ ✓
$= 1770$ ✓

Calculator Free

2. [7 marks: 1, 2, 2, 2]

Determine the value of integer r (where $r \geq 0$) in each of the following equations:

(a) $\binom{12}{8} = \binom{12}{r}$

$8 + r = 12$
$r = 4$ ✓

(b) $\binom{30}{r} = \binom{30}{r+4}$

$r + r + 4 = 30$ ✓
$r = 13$ ✓

(c) $\binom{25}{2r} - \binom{25}{r-2} = 0$

$2r + r - 2 = 25$ ✓
$r = 9$ ✓

(d) $\binom{r}{0} + \binom{r}{1} + \binom{r}{2} + \binom{r}{3} + \binom{r}{4} = 2^r$

Sum of binomial coefficients $\sum_{k=0}^{r}\binom{r}{k} = 2^r$ ✓
Hence, $r = 4$. ✓

3. [4 marks: 1, 3]

Expand completely each of the following:

(a) $(1 + x)^5$

$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ ✓

(b) $(2 - x)^4$

$2^4 + 4(2^3)(-x) + 6(2^2)(-x)^2 + 4(2)(-x)^3 + (-x)^4$ ✓✓
$= 16 - 32x + 24x^2 - 8x^3 + x^4$ ✓

Calculator Free

4. [13 marks: 1, 4, 5, 3]

Consider the expansion for $\left(x^2 - \frac{2}{x}\right)^{12}$ in *descending* powers of x.

(a) How many terms are there in this expansion?

> Number of terms = 12 + 1 = 13. ✓

(b) Find the third term in this expansion.

> Third term = $\binom{12}{2} \times (x^2)^{10} \times \left(\frac{-2}{x}\right)^2$ ✓✓
>
> $= \frac{12\times11}{2\times1} \times x^{20} \times \frac{4}{x^2}$ ✓
>
> $= 264\,x^{18}$ ✓

(c) Find a mathematical expression for the coefficient of the term in $\frac{1}{x^{12}}$.

> General term = $\binom{12}{r} \times (x^2)^{12-r} \times \left(\frac{-2}{x}\right)^r$ ✓✓
>
> Power for term = $2(12 - r) - r$
> $= 24 - 3r$
>
> Hence, $24 - 3r = -12$ ✓
> $r = 12$
>
> Coefficient of required term = $\binom{12}{12} \times (-2)^{12}$ ✓✓
> $= 4096$

(d) Find a mathematical expression for the term independent of x.

> Power for term = $24 - 3r$
> Hence, $24 - 3r = 0$
> $r = 8$ ✓
>
> Required term = $\binom{12}{8} \times (-2)^8$ ✓✓

Calculator Free

5. [5 marks: 1, 2, 2]

Amy has a collection of 18 fluoro pens in her pink box and 24 fluoro pens in her blue box. Write mathematical expressions for the number of ways Amy can pick:

(a) three pens from her pink box.

> No. of ways = ${}^{18}C_3$ ✓

(b) three pens from the pink box and four pens from the blue box.

> No. of ways = ${}^{18}C_3 \times {}^{24}C_4$ ✓✓

(c) a dozen pens from both boxes.

> No. of ways = ${}^{42}C_{12}$ ✓✓

6. [10 marks: 1, 2, 2, 3, 2]

A committee of 9 people is to be selected from 10 Labor, 8 Liberal and 5 Green politicians. Write mathematical expressions for the number of different ways the committee can be selected if:

(a) there are no restrictions.

> No. of ways = ${}^{23}C_9$ ✓

(b) all three political parties are equally represented.

> No. of ways = ${}^{10}C_3 \times {}^{8}C_3 \times {}^{5}C_3$ ✓✓

(c) there are no Greens.

> No. of ways = ${}^{18}C_9$ ✓✓

(d) the Liberal representatives are in the majority.

> No. of ways = ${}^{15}C_4 \times {}^{8}C_5 + {}^{15}C_3 \times {}^{8}C_6 + {}^{15}C_2 \times {}^{8}C_7 + {}^{15}C_1 \times {}^{8}C_8$ ✓✓✓

(e) the Labor husband and wife pair, Alex and Alice, cannot be in the same committee.

> No. of ways = ${}^{23}C_9 - {}^{2}C_2 \times {}^{21}C_7$ ✓✓

Calculator Assumed

7. [19 marks: 1, 3, 3, 4, 4, 4]

Consider the digits 0 to 9 inclusive and all the letters of the alphabet. Ten characters consisting of digits and letters are chosen. Determine the number of ways of choosing:

(a) all the even numbers and all the vowels.

Number of ways = 1 ✓

(b) any six digits and any four letters.

Number of ways = $^{10}C_6 \times {}^{26}C_4$ ✓✓
= 3 139 500 ✓

(c) exactly four vowels.

Number of ways = $^{5}C_4 \times {}^{31}C_6$ ✓✓
= 3 681 405 ✓

(d) at least four odd digits.

Number of ways = $^{5}C_4 \times {}^{31}C_6 + {}^{5}C_5 \times {}^{31}C_5$ ✓✓✓
= 3 681 405 + 169 911
= 3 851 316 ✓

(e) four vowels and four odd digits.

Number of ways = $^{5}C_4 \times {}^{5}C_4 \times {}^{26}C_2$ ✓✓✓
= 8 125 ✓

(f) four vowels or four odd digits.

Number of ways = N(4 vowels) + N(4 odd digits) – N(4 vowels & 4 odd digits) ✓✓✓
= 3 681 405 + 3 681 405 – 8 125
= 7 354 685 ✓

21 Probability I

Calculator Free

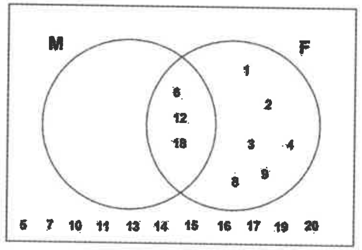
1. [9 marks: 1, 2, 2, 2, 2]

Let the universal set U be the set of all integers between 1 and 20 inclusive.
Let M: $\{x \mid x$ is a multiple of 6$\}$ and F: $\{x \mid x$ is a factor of 72$\}$.

(a) List all the elements in F.

F = {1, 2, 3, 4, 6, 8, 9, 12, 18} ✓

Use a Venn Diagram or a two-way table to the answer the following questions.



	M	$\overline{M}$
F	6, 12, 18	1, 2, 3, 4, 8, 9
$\overline{F}$		5, 7, 10, 11, 13, 14, 15, 16, 17, 19, 20

(b) How many multiples of 6 are not factors of 72?

Number = 0 ✓✓

(c) How many integers are either multiples of 6 or are factors of 72?

Number = 9 ✓✓

(d) Find the probability that a randomly chosen integer:
(i) is a neither a multiple of 6 nor a factor of 72.

Probability = $\frac{11}{20}$ ✓✓

(ii) is a multiple of 6 given that it is a factor of 72.

Probability = $\frac{3}{9}$ ✓✓

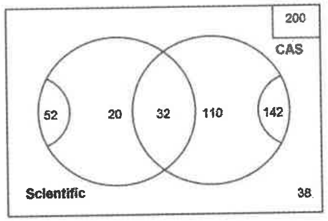
Calculator Assumed

2. [8 marks: 2, 2, 2, 2]

The Mathematics department at a school conducted a random survey involving 200 students.

- 38 students did not have any calculator (scientific or CAS) with them
- 142 students had a CAS calculator with them
- 52 students had a scientific calculator with them

Use a Venn Diagram or a two-way table to the answer the following questions.



Total	CAS	$\overline{CAS}$	
SC	32	20	52
$\overline{SC}$	110	38	148
Total	142	58	200

(a) Find the probability that a student chosen at random had only a CAS calculator.

Probability = $\frac{110}{200}$ ✓✓

(b) Find the probability that a student chosen at random had both a CAS calculator as well as a scientific calculator.

Probability = $\frac{32}{200}$ ✓✓

(c) Find the probability that a student chosen at random had either a CAS calculator or a scientific calculator.

Probability = $\frac{162}{200}$ ✓✓

(d) Find the probability that a student selected from those who had at least one type of calculator had both types of calculator.

Probability = $\frac{32}{110+32+20}$

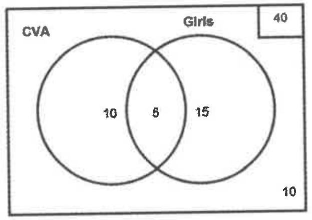
$= \frac{32}{162}$ ✓✓

Calculator Assumed

3. [8 marks: 2, 2, 2, 2]

In a group of 40 students, there are 10 boys who are colour vision impaired (CVA) and 15 girls who are not colour vision impaired. There are as many boys who are not colour vision impaired as there are boys who are colour vision impaired.

Use a Venn Diagram or a two-way table to the answer the following questions.



	Girls	$\overline{\text{Girls}}$	Total
CVA	5	10	15
$\overline{CVA}$	15	10	25
Total	20	20	40

(a) How many girls were colour vision impaired?

No.of girls = 5 ✓✓

(b) A student is randomly chosen from this group. Find the probability that this student is a girl.

Probability = $\frac{20}{40}$ ✓✓

(c) A student is randomly chosen from this group. Find the probability that this student is either a girl or is colour vision impaired.

Probability = $\frac{30}{40}$ ✓✓

(d) A student is randomly chosen from this group. Given that this student is either a girl or is colour vision impaired, find the probability that this student is colour vision impaired.

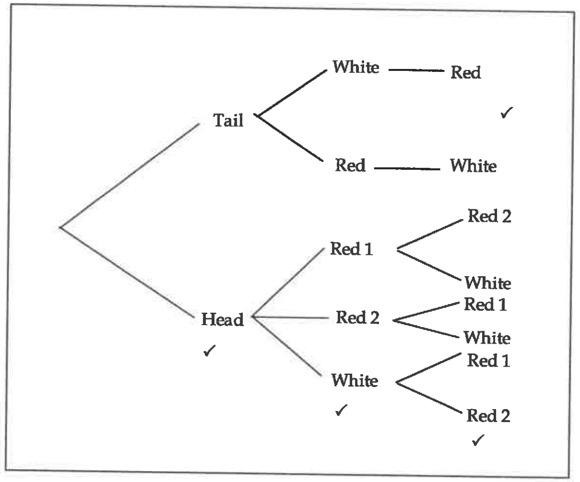
Probability = $\frac{15}{30}$ ✓✓

Calculator Assumed

4. [10 marks: 4; 1, 1, 2, 2]

Box A contains two red balls and a white ball. Box B contains one white ball and a red ball. A coin is tossed. If a tail appears, 2 balls are drawn without replacement from box B. If a head appears, 2 balls are drawn without replacement from box A.

(a) Use a tree diagram to display all the possible outcomes.



(b) If each outcome is equally likely, find the probability that:

(i) exactly two red balls are chosen.

Probability = $\frac{2}{8}$ ✓

(ii) exactly one white ball is chosen.

Probability = $\frac{6}{8}$ ✓

(iii) exactly two red balls are chosen given that box A was chosen.

Probability = $\frac{2}{6}$ ✓✓

(iv) box A was chosen given that exactly one white ball was chosen.

Probability = $\frac{4}{6}$ ✓✓

Calculator Assumed

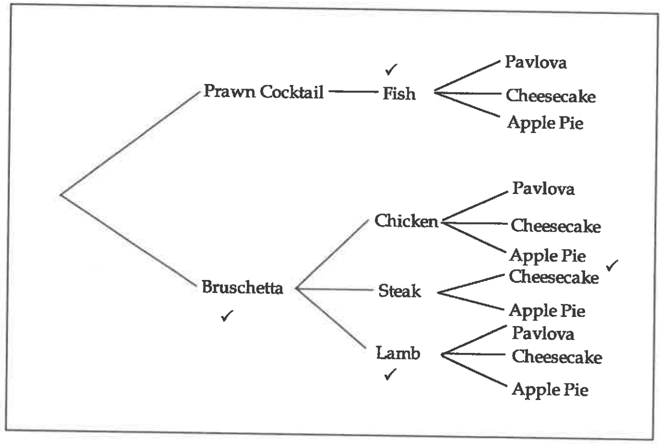
5. [10 marks: 4, 6]

Allie's Café offers the following menu:

Appetisers: *Prawn Cocktail or Bruschetta*
Main Meal: *Fish or Chicken or Steak or Lamb*
Dessert: *Pavlova or Cheesecake or Apple Pie with Ice Cream*

Jenny orders an appetiser, one item for the main meal and one dessert. Jenny will not have Steak with Pavlova and must have Prawn Cocktail with Fish and Fish only with Prawn Cocktail.

(a) Display Jenny's possible appetiser/main meal/dessert combinations in a clearly labelled tree diagram.



(b) If each of Jenny's possible appetiser/main meal/dessert combination are equally likely, find the probability that Jenny:

(i) chose chicken as a main meal.

Probability = $3/11$ ✓

(ii) did not choose cheesecake as a dessert.

Probability = $7/11$ ✓

(iii) chose cheesecake given that she chose chicken.

Probability = $1/3$ ✓✓

(iv) chose chicken given that she chose cheesecake.

Probability = $1/4$ ✓✓

Calculator Assumed

6. [10 marks: 1, 3, 3, 3]

A red box has four books and a blue box has eight books. All books are different. A total of five books are chosen from these two boxes.

(a) In how many ways can this be done?

No. of ways = ${}^{12}C_5 = 792$ ✓

(b) What is the probability that all the books from the red box are chosen?

Prob. $= \dfrac{{}^4C_4\,{}^8C_1}{{}^{12}C_5}$ ✓✓

$= \dfrac{8}{792}$ (or $\dfrac{1}{99}$) ✓

(c) What is the probability that at least one of the books chosen is from the red box?

Prob. = 1 – P(none from the red box)
= 1 – P(all five are from the blue box)

$= 1 - \dfrac{{}^4C_0\,{}^8C_5}{{}^{12}C_5}$ ✓✓

$= 1 - \dfrac{56}{792} = \dfrac{736}{792}$ (or $\dfrac{92}{99}$) ✓

(d) What is the probability that more books from the red box are chosen?

Prob. = P(4 from red box & 1 from blue box)
+ P(3 from red box & 2 from blue box)

$= \dfrac{8}{792} + \dfrac{{}^4C_3\,{}^8C_2}{{}^{12}C_5}$ ✓✓

$= \dfrac{8}{792} + \dfrac{112}{792} = \dfrac{120}{792}$ (or $\dfrac{5}{33}$) ✓

Calculator Assumed

7. [15 marks: 3, 3, 3, 3, 3]

Consider the digits 0 to 9 inclusive and all the letters of the English Roman alphabet. Twelve characters consisting of digits and letters are chosen.

(a) What is the probability that all the characters chosen are letters?

Prob. $= \dfrac{{}^{10}C_0\,{}^{26}C_{12}}{{}^{36}C_{12}}$ ✓✓

$= 0.007\,716 \approx 0.0077$ ✓

(b) What is the probability that all the digits are chosen?

Prob. $= \dfrac{{}^{10}C_{10}\,{}^{26}C_2}{{}^{36}C_{12}}$ ✓✓

$= 0.000\,000\,259\,65 \approx 0$ ✓

(c) Given that all the characters chosen are letters, what is the probability that all the vowels are chosen?

Prob. $= \dfrac{{}^5C_5\,{}^{21}C_7}{{}^{26}C_{12}}$ ✓✓

$= 0.012\,040 \approx 0.00120$ ✓

(d) What is the probability that no vowels or even digits were chosen?

Prob. $= \dfrac{{}^{26}C_{12}}{{}^{36}C_{12}}$ ✓✓

$= 0.007\,716 \approx 0.0077$ ✓

(e) What is the probability that at least one vowel or even digit was chosen?

Prob. $= 1 - \dfrac{{}^{26}C_{12}}{{}^{36}C_{12}}$ ✓✓

$= 0.992\,284 \approx 0.9923$ ✓

Calculator Assumed

8. [11 marks: 1, 1, 1, 2, 3, 3]

Last year, Malcolm was late to school on average, 5 days out of 100 days. Write mathematical expressions (but do not evaluate) for the probability that in a school week of 5 days, Malcolm is:

(a) late only on the first day.

Prob. $= 0.05 \times 0.95^4$ ✓

(b) late on the first three days.

Prob. $= 0.05^3$ ✓

(c) late only on the first three days.

Prob. $= 0.05^3 \times 0.95^2$ ✓

(d) late only on exactly three days.

Prob. $= {}^5C_3 \times 0.05^3 \times 0.95^2$ ✓✓

(e) late on at least three days.

Prob. $= {}^5C_3 \times 0.05^3 \times 0.95^2$ ✓
$+ {}^5C_4 \times 0.05^4 \times 0.95$ ✓
$+ 0.05^5$ ✓

(f) late only the first and the fifth day given that he was late on exactly two days in the school week.

Prob. $= \dfrac{0.05^2 \times 0.95^3}{{}^5C_2 \times 0.05^2 \times 0.95^3}$ ✓✓
$= \dfrac{1}{{}^5C_2}$ ✓

Calculator Assumed

9. [9 marks: 1, 2, 3, 3] [TISC]

Zico practices kicking a soccer ball from the penalty spot. From previous practices, on average, he scores 70 goals from 100 attempts.

(a) Find the probability that Zico's first two kicks do not score goals.

Prob. $= 0.3 \times 0.3$
$= 0.09$ ✓

(b) Find the probability that Zico's first kick scores a goal but the next two kicks do not score goals.

Prob. $= 0.7 \times 0.3 \times 0.3$ ✓
$= 0.063$ ✓

If Zico has 10 kicks of the ball from the penalty spot, find the probability that,

(c) he scores exactly 5 goals.

Prob. $= {}^{10}C_5 \times (0.7)^5 \times (0.3)^5$ ✓✓
$= 0.1029$ ✓

(d) he scores goals only on the first, fifth, seventh and ninth kick.

Prob. $= (0.7)^4 \times (0.3)^6$ ✓✓
$= 0.000\,175$ ✓

Calculator Assumed

10. [11 marks: 2, 3, 3, 3]

The *Collett Boat Company* has a fleet of three boats. From Company records for the last two years, the *Jupiter* is chosen by 55% of customers, the *Venus* by 28% of customers and the *Mars* by the remaining customers. The probabilities that each boat breaks down during a two-hour trip are *Jupiter* 0.2; *Venus* 0.15; *Mars* 0.3.

(a) If all three boats are out on hire for a two-hour trip, find the probability that:
(i) none breaks down.

Prob. $= 0.8 \times 0.85 \times 0.7$ ✓
$= 0.476$ ✓

(ii) the *Mars* and one other boat in this fleet breaks down.

Prob. $= 0.3 \times 0.2 \times 0.85$ ✓
$+ 0.3 \times 0.8 \times 0.15$ ✓
$= 0.087$ ✓

(b) Only one boat is out on hire for a two-hour trip. What is the probability that it will break down.

Prob. $= 0.55 \times 0.2$
$+ 0.28 \times 0.15$
$+ 0.17 \times 0.3$ ✓✓
$= 0.203$ ✓

(c) News comes through that the one boat out on hire has broken down. What is the probability that it is the *Jupiter*?

P(Jupiter | One boat broken down)

$$= \frac{\text{P(Jupiter} \cap \text{One boat broken down)}}{\text{P(One boat broken down)}}$$

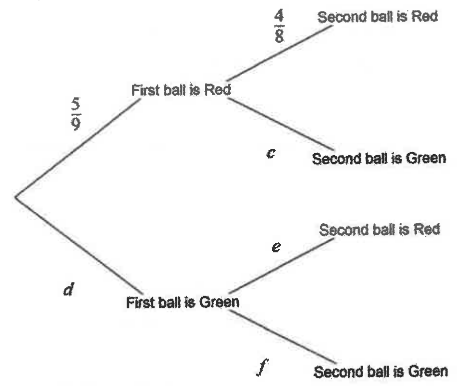
$$= \frac{0.55 \times 0.2}{0.203}$$ ✓✓

$= 0.5419$ ✓

Calculator Assumed

11. [9 marks: 4, 2, 3]

A box has five red balls and four green balls. Two balls are drawn without replacement from this box. The tree diagram indicates the associated outcomes and the corresponding probabilities.



(a) State the probability values *c*, *d*, *e* and *f*.

$c = \frac{4}{8}$ ✓ $d = \frac{4}{9}$ ✓ $e = \frac{5}{8}$ ✓ $f = \frac{3}{8}$ ✓

(b) Find the probability that both balls are red.

Prob. = P(red ∩ red)
$= \frac{5}{9} \times \frac{4}{8}$ ✓
$= \frac{5}{18}$ ✓

(c) Find the probability that both balls are of the same colour. Show clearly how you obtained your answer.

Prob. = P(red ∩ red) + P(green ∩ green) ✓
$= \frac{5}{18} + \frac{4}{9} \times \frac{3}{8}$ ✓
$= \frac{4}{9}$ ✓

Calculator Assumed

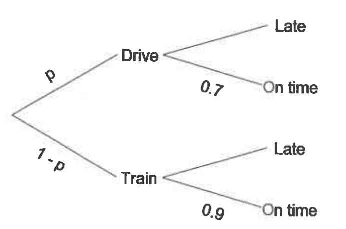
12. [10 marks: 2, 4, 4] [TISC]

Chin either drives to work or takes a train to work. The probability that he is on time for work is 0.86. The probability that he is late for work given that he drives to work is 0.3. The probability that he is on time for work given that he takes a train is 0.9.

(a) Find the probability that he is on time for work given that he drives to work.

Prob. = 1 − 0.3 ✓
= 0.7 ✓

(b) Find the probability that he drives to work.



Let p: prob. Chin drives to work.

P(On time) = 0.86
Hence:
$p(0.7) + (1-p)(0.9) = 0.86$ ✓✓✓
$\Rightarrow \; p = 0.2$ ✓

(c) Given that he was late for work, what is the probability that he took the train to work.

$$P(\text{Train} \mid \text{Late}) = \frac{P(\text{Train} \cap \text{Late})}{P(\text{Late})} = \frac{0.8 \times 0.1}{1 - 0.86} = \frac{4}{7}$$ ✓✓✓ ✓

Calculator Assumed

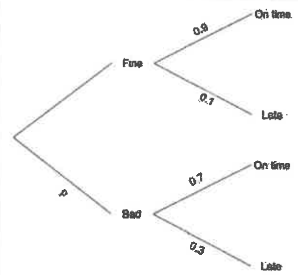
13. [9 marks: 1, 5, 3]

At a certain airport, the probability that a plane takes off on time given that weather conditions are fine is 0.9. The probability that a plane takes off on time given that weather conditions are bad is 0.7. The probability of weather conditions being fine or the plane taking off on time is 0.955.

(a) Find the probability that a plane does not take off on time given that weather conditions were bad.

Prob. = 1 − 0.7 = 0.3 ✓

(b) Find the probability of a plane taking off on time in fine weather conditions.



Let p: P(weather conditions is bad)

P(Bad weather ∩ late) = 1 − 0.955 = 0.045 ✓

Hence, $p \times 0.3 = 0.045$ ✓
$p = 0.15$ ✓

Hence P(fine weather ∩ on time) = (1 − 0.15) × 0.9 ✓
= 0.765 ✓

(c) Find the probability of weather conditions being fine given that a plane took off on time.

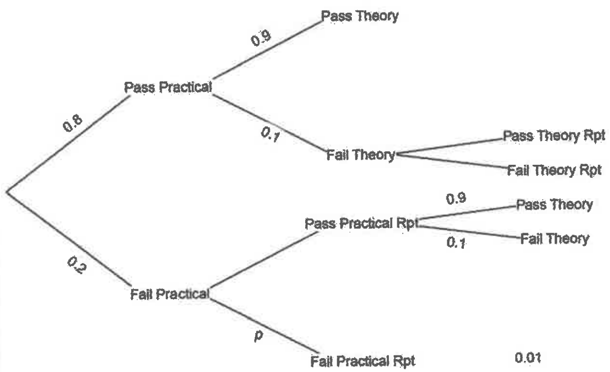
$$P(\text{fine} \mid \text{on time}) = \frac{P(\text{fine} \cap \text{on time})}{P(\text{on time})} = \frac{0.765}{0.765 + 0.15 \times 0.7} = 0.8793$$ ✓✓ ✓

Calculator Assumed

14. [11 marks: 5, 6]

To be awarded a pass for a course, students are required to first pass a practical examination and then a theory examination. A student is only allowed one failure. A student that fails two examinations automatically fails the course. In all cases, a student is allowed to repeat a failed examination only once. The probability that a student will pass the practical examination on the first attempt is 0.8. The probability that a student will pass the theory examination given that the student has passed the practical examination is 0.9, even if the student failed the practical examination in the first attempt. The probability that a student fails outright (fails the practical examination on the first and second attempt) is 0.01 and the probability that a student fails the course is 0.045.

(a) Find the probability that a randomly chosen student fails the entire course given that the student failed the practical examination on the first attempt.



Let p: P(Fail prac rpt | Fail prac)

P(Fail outright) = P(Fail prac and Fail prac rpt) = 0.01

P(Fail prac and Fail prac rpt) = P(Fail prac) × P(Fail prac rpt | Fail prac) ✓

$0.01 = 0.2 \times p$

$p = 0.05$ ✓

$$\text{P(Fails course | Failed prac)} = \frac{\text{P(Fails course} \cap \text{Failed prac)}}{\text{P(Failed prac)}}$$

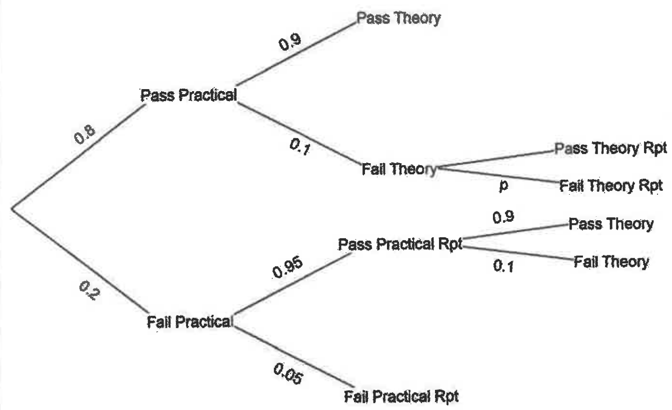
$$= \frac{0.2\times0.95\times0.1+0.01}{0.2} \quad ✓✓$$

$= 0.145$ ✓

OR P(Fails course | Failed prac) = $0.05 + 0.95 \times 0.1 = 0.145$

Calculator Assumed

14. (b) Find the probability that a randomly chosen student passes the course given that the student had to repeat an examination.



Let p: P(Fail Theory Rpt | Fail Theory)

P(Fail course) = 0.045

Hence: $(0.8 \times 0.1 \times p) + (0.2 \times 0.95 \times 0.1) + 0.01 = 0.045$ ✓

$\Rightarrow \quad p = 0.2$ ✓

$$\text{P(pass course | repeat an exam)} = \frac{\text{P(passed course} \cap \text{repeated an exam)}}{\text{P(repeated an exam)}}$$

$$= \frac{0.8\times0.1\times0.8+0.2\times0.95\times0.9}{0.8\times0.1+0.2} \quad ✓✓✓$$

$= 0.8393$ ✓

22 Probability II

Calculator Free

1. [5 marks: 3, 2]

Given that $P(\overline{A \cup B}) = 0.2$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$:

(a) find $P(B \mid A)$.

$P(B \mid A) = \frac{0.2}{0.6}$ ✓✓

$= \frac{1}{3}$ ✓

	A	A′	
B	0.2	0.2	0.4
B′	0.4	0.2	0.6
	0.6	0.4	1

(b) determine with reasons if the events A and B are independent.

$P(B) = 0.4$

But $P(B \mid A) = \frac{1}{3}$

Hence, $P(B) \neq P(B \mid A)$. ✓

Therefore, A and B are not independent. ✓

2. [6 marks]

Given that $P(A) = 0.4$, $P(C \mid A) = 0.3$, $P(C \mid \bar{A}) = 0.2$, find $P(A \mid C)$.

$P(C \mid A) = \frac{k}{0.4} = 0.3$ ✓

$\Rightarrow k = 0.12$ ✓

$P(C \mid A') = \frac{m}{0.6} = 0.2$ ✓

$\Rightarrow m = 0.12$ ✓

Hence, $P(C) = k + m = 0.24$ ✓

$\Rightarrow P(A \mid C) = \frac{0.12}{0.24}$

$= 0.5$ ✓

	A	A′	
C	k	m	
C′			
	0.4	0.6	1

Calculator Free

3. [8 marks: 2, 2, 4]

It is known that $P(A) = 0.6$ and $P(B) = 0.3$. Find:

(a) $P(B \mid A)$ given that A and B are mutually exclusive.

$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$ ✓

$= 0$ as $P(B \cap A) = 0$ ✓

(b) $P(A \mid B)$ given that A and B are independent.

$P(A \mid B) = P(A)$ ✓

$= 0.6$ ✓

(c) $P(B \mid A)$ given that $P(A \mid B) = 0.2$.

$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$ ✓

$= \frac{P(B) \times P(A \mid B)}{P(A)}$ ✓

$= \frac{0.3 \times 0.2}{0.6}$ ✓

$= 0.1$ ✓

Calculator Free

4. [8 marks: 4, 2, 2]

Given that $P(A') = \frac{3}{10}$, $P(B|A) = \frac{2}{7}$ and $P(A|B) = \frac{1}{2}$:

(a) find P(B).

$P(B|A) = \frac{k}{0.7} = \frac{2}{7}$ ✓

$\Rightarrow k = 0.2$ ✓

$P(A|B) = \frac{0.2}{m} = 0.5$ ✓

$\Rightarrow m = 0.4$ ✓

Hence, $P(B) = m = 0.4$

	A	A'	
B	k		m
B′			
	0.7	0.3	1

(b) find $P(A' \cap B')$.

$P(A' \cap B') = 0.1$ ✓✓

	A	A'	
B	0.2	0.2	0.4
B′		0.1	
	0.7	0.3	1

(c) determine with reasons if A and B are independent.

$P(B) = 0.4$

But $P(B|A) = \frac{2}{7}$

Hence, $P(B) \neq P(B|A)$. ✓

Therefore, A and B are not independent. ✓

Calculator Assumed

5. [5 marks: 1, 1, 3]

$P(A) = 0.3$, $P(B') = 0.4$ and $P(A \cap B) = k$:

(a) find in terms of k,

(i) $P(A \cap B')$.

$P(A \cap B') = 0.3 - k$ ✓

(ii) $P(A' \cap B)$.

$P(A' \cap B) = 0.6 - k$ ✓

	A	A′	
B	k	$0.6 - k$	0.6
B′	$0.3 - k$		0.4
	0.3	0.7	

(b) find k given that $P(\overline{A \cup B}) = 0.18$.

$(0.3 - k) + 0.18 = 0.4$ ✓✓

$\Rightarrow k = 0.08$ ✓

6. [4 marks]

Given that $P(A) = 0.5$, $P(B) = 0.8$ and $P(\overline{A} \cap \overline{B}) = 0.05$ and that the events A and B are independent, determine if these results are consistent with the rules of probability. Justify your answer.

Since A and B are independent:

$P(A \cap B) = P(A) \times P(B) = 0.5 \times 0.8 = 0.4$ ✓

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.5 + 0.8 - 0.4$
$= 0.9$ ✓

But $P(\overline{A} \cap \overline{B}) = 0.05$.

$\Rightarrow P(A \cup B) = 1 - 0.05 = 0.95 \neq 0.9$ ✓

Hence, these results are not consistent with the rules of probability. ✓

Calculator Assumed

7. [10 marks: 1, 2, 4, 3]

Given that P(A) = p + 0.2 and P(B) = p + 0.3 and P(A ∩ B) = p, calculate the value of p if:

(a) A and B are mutually exclusive events.

$p = 0$ ✓

(b) P(A ∪ B) = 0.6.

P(A ∪ B) = 0.6
⇒ P(A) + P(B) − P(A ∩ B) = 0.6
$(p + 0.2) + (p + 0.3) - p = 0.6$ ✓
$p = 0.1$ ✓

(c) A and B are independent events.

P(A ∩ B) = P(A) × P(B)
$p = (p + 0.2)(p + 0.3)$ ✓
$p = p^2 + 0.5p + 0.06$
$p^2 - 0.5p + 0.06 = 0$ ✓
$p = 0.2$ or 0.3 ✓✓

(d) P(A | B) = 0.4.

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$0.4 = \frac{p}{p+0.3}$ ✓

$0.4p + 0.12 = p$ ✓
$0.6p = 0.12$
$p = 0.2$ ✓

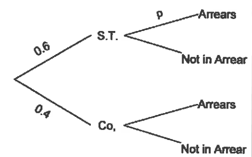
Calculator Assumed

8. [9 marks: 4, 3, 2]

A supply company checked its accounts and found that 8% of accounts were in arrears. 60% of the accounts were for sole traders while the other 40% were for companies. Only 2% of accounts were both in arrears and were accounts of sole traders.

(a) Find the probability of an account not being in arrears belonging to a sole trader.

Let P(Arrears | Sole Trader) = p



P(Arrears & Sole Trader) = 0.02
Hence:
$0.6 \times p = 0.02$ ✓✓
$\Rightarrow p = \frac{1}{30}$ ✓

Therefore P(S.T. & Not in Arrears)
$= 0.6 \times (1 - \frac{1}{30})$
$= \frac{29}{50}$ ✓

(b) Of accounts in arrears, find the probability of the account belonging to a sole trader.

$P(S.T.|Arrears) = \frac{P(S.T. \cap Arrears)}{P(Arrears)}$
$= \frac{0.02}{0.08}$ ✓✓
$= \frac{1}{4}$ ✓

(c) Is the account status independent of the type of trader? Justify your answer.

P(Arrears) = 0.08
But P(Arrears | Sole Trader) = $\frac{1}{30}$
Hence, P(Arrears) ≠ P(Arrears | Sole Trader) ✓
Therefore, account status is not independent of type of trader. ✓

Calculator Assumed

9. [7 marks: 1, 2, 2, 2] [TISC]

John drives to work each weekday morning. The route he takes passes through a set of traffic lights where he either has to stop at the lights or move through without stopping.

- If he has to stop at the lights, the probability that he will be late for work is 0.7.
- If he does not have to stop at the lights, the probability that he will be late for work is 0.2.

Overall, the probability that John will be late for work is 0.25.

(a) Find the probability that he will not be late for work if he did not have to stop at the traffic lights.

Prob. $= 1 - 0.2 = 0.8$ ✓

(b) Find the probability that John has to stop at the traffic lights.

Let p: probability that John has to stop at the lights.

P(late for work) = 0.25
$p \times 0.7 + (1-p) \times 0.2 = 0.25$ ✓
$p = 0.1$ ✓

(c) Determine with reasons if John being late is independent of whether he has to stop at the lights.

P(John late | John stops at the lights) = 0.7
P(John is late) = 0.25 ≠ 0.7 ✓
Hence, John being late is ***not*** independent of whether he has to stop at the lights. ✓

(d) Find the probability that John had to stop at the lights given that he was late for work.

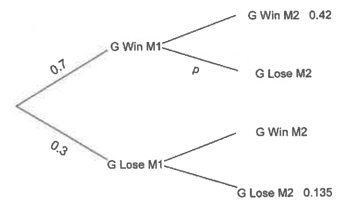
$$P(\text{John stops at the lights} \mid \text{John Late}) = \frac{P(\text{John late and had to stop at the lights})}{P(\text{John late for work})} = \frac{0.1 \times 0.7}{0.25} = \frac{7}{25}$$ ✓✓

Calculator Assumed

10. [10 marks: 2, 3, 3, 2] [TISC]

England and Germany play a series of two soccer matches. Each match does not end in a draw. The probability that Germany will win both matches is 0.42 and the probability that Germany will lose both matches is 0.135. The probability that Germany will win the first match is 0.7.

(a) Find the probability that England wins exactly one of the matches.



Prob. $= 1 - 0.42 - 0.135 = 0.445$ ✓✓

(b) Find the probability that England win the second match.

$P(\text{G Win M2} \mid \text{G Win M1}) = \frac{0.42}{0.7} = 0.6$
$\Rightarrow p = 1 - 0.6 = 0.4.$ ✓
Hence, P(E wins match 2) $= 0.7 \times 0.4 + 0.135$ ✓
$= 0.415$ ✓

(c) Find the probability that Germany lose the first match given that they won the second match.

$$P(\text{G Lose M1} \mid \text{G Win M2}) = \frac{P(\text{G Lose M1} \cap \text{G Win M2})}{P(\text{G Win M2})} = \frac{1 - 0.42 - 0.415}{1 - 0.415}$$ ✓✓
$= 0.2821$ ✓

(d) Determine with reasons if the events that Germany win match one and the event that Germany win match two are independent.

P(G Win M1) × P(G Win M2) $= 0.7 \times (1 - 0.415) = 0.4095$ ✓
But P(G Win M1 ∩ G Win M2) = 0.42.
Since P(G Win M1) × P(G Win M2) ≠ P(G Win M1 ∩ G Win M2), G Win M1 and G Win M2 are not independent. ✓

23 Indices

Calculator Free

1. [12 marks: 2, 2, 2, 3, 3]

Simplify each of the following leaving answers with positive indices:

(a) $\dfrac{16x^2y^3}{12x}$

$\dfrac{4xy^3}{3}$ ✓✓

(b) $\left(\dfrac{3x^2}{y}\right)^{-2}$

$\dfrac{3^{-2}x^{-4}}{y^{-2}} = \dfrac{y^2}{9x^4}$ ✓ ✓

(c) $\dfrac{81x^{-4}y^5}{36x^2y^{-2}}$

$\dfrac{9y^7}{4x^6}$ ✓✓

(d) $\dfrac{(4x^2y^4z)^{\frac{1}{2}}}{xy^{-1}\sqrt{z}}$

$\dfrac{2xy^2z^{\frac{1}{2}}}{xy^{-1}z^{\frac{1}{2}}} = 2y^3$ ✓✓ ✓

(e) $\dfrac{x^3y^2}{z^2y^{-1}} \times \dfrac{y\sqrt{z}}{x^{-1}}$

$\dfrac{x^3y^2}{z^2y^{-1}} \times \dfrac{yz^{\frac{1}{2}}}{x^{-1}} = \dfrac{x^4y^4}{z^{\frac{3}{2}}}$ ✓✓ ✓

Calculator Free

2. [13 marks: 2, 2, 2, 3, 4]

Simplify each of the following, leaving answers with positive indices:

(a) $\dfrac{5^{n+1} - 5^n}{8}$

$\dfrac{5^n(5-1)}{8} = \dfrac{5^n}{2}$ ✓ ✓

(b) $\dfrac{5^{n+2} - 5^{n+1}}{2(5^{n+1})}$

$\dfrac{5^{n+1}(5-1)}{2(5^{n+1})} = 2$ ✓ ✓

(c) $\dfrac{7^{n-1} + 7^n}{4 \times 7^{n-1}}$

$\dfrac{7^{n-1}(1+7)}{4 \times 7^{n-1}} = 2$ ✓ ✓

(d) $\dfrac{3^{2n} + 3^n}{3^{n+1} + 3}$

$\dfrac{3^n(3^n + 1)}{3(3^n + 1)} = 3^{n-1}$ ✓✓ ✓

(e) $\dfrac{2^{n+1} + 8}{3(2^n) + 12}$

$\dfrac{2^{n+1} + 2^3}{3(2^n) + 3(2^2)}$ ✓

$= \dfrac{2^3(2^{n-2} + 1)}{3(2^2)[2^{n-2} + 1]}$ ✓✓

$= \dfrac{2}{3}$ ✓

Calculator Free

3. [12 marks: 2, 2, 2, 3, 3]

Solve for t.

(a) $3^{2t+1} = 81$

$3^{2t+1} = 3^4 \Rightarrow 2t + 1 = 4$ ✓
Hence, $t = \frac{3}{2}$ ✓

(b) $4^{1-t} = 32$

$(2^2)^{1-t} = 2^5 \Rightarrow 2 - 2t = 5$ ✓
Hence, $t = -\frac{3}{2}$ ✓

(c) $5^{2+t} = \frac{1}{125}$

$5^{2+t} = 5^{-3}$ ✓
$2 + t = -3$
Hence, $t = -5$. ✓

(d) $5^t \times 25^{t-1} = 0.04$

Rewrite equation as: $5^t \times (5^2)^{t-1} = 5^{-2}$ ✓
$5^{t+2t-2} = 5^{-2}$ ✓
Hence, $3t - 2 = -2 \Rightarrow t = 0$ ✓

(e) $\dfrac{2^{2t+1}}{2^{1-t}} = 4$

Rewrite equation as: $2^{2t+1-(1-t)} = 2^2$ ✓
$2^{3t} = 2^2$ ✓
Hence, $\Rightarrow t = \frac{2}{3}$ ✓

Calculator Free

4. [3 marks]

Solve for x in $(2^x)^2 + 2(2^x) - 8 = 0$.

Hint: Replace 2^x with y.

Equation becomes: $y^2 + 2y - 8 = 0$
$(y + 4)(y - 2) = 0$ ✓
$y = 2$ or -4 ✓
Hence: $2^x = 2 \Rightarrow x = 1$ ✓
or $2^x = -4$ (not possible)

5. [5 marks]

Solve for x, $3^{2x+1} + 8(3^x) - 3 = 0$.

Rewrite equation as: $3(3^x)^2 + 8(3^x) - 3 = 0$
Let $y = 3^x$.
Hence, equation becomes:
$3(y)^2 + 8(y) - 3 = 0$ ✓
$(3y - 1)(y + 3) = 0$
Hence, $y = \frac{1}{3}$ or -3 ✓✓

Therefore: $3^x = \frac{1}{3}$ or $3^x = -3$

For $3^x = \frac{1}{3}$, $3^x = 3^{-1} \Rightarrow x = -1$ ✓

For $3^x = -3$, there is no solution for x. ✓

24 Arithmetic Progressions

Calculator Free

1. [4 marks]

An arithmetic sequence has first term 10 and common difference 8. State the recursive rule and general rule for this sequence.

Recursive rule is	$T_{n+1} = T_n + 8$ with $T_1 = 10$ ✓ ✓
General Rule is	$T_n = 2 + 8n$ ✓ ✓

2. [5 marks: 3, 2]

The terms of a sequence are defined by $T_{n+1} - T_n + 20 = 0$ with $T_1 = 100$

(a) Show that this sequence is an arithmetic sequence.

Rewrite recursive rule as $T_{n+1} - T_n = -20$ with $T_1 = 100$ ✓
Hence, difference between any two consecutive terms is constant. ✓
Hence, sequence is an arithmetic sequence. ✓

(b) How many positive terms are there in this sequence?

Terms are 100, 80, 60, 40, 20, 0, –20 .....
Hence, there are 5 positive terms. ✓✓

3. [4 marks]

An arithmetic sequence is described by the rule $T_n = 150 - 4n$, where $n = 1, 2, 3, 4, 5, \ldots$. Find the first three terms of the sequence. Hence, state the recursive rule for this sequence.

$T_1 = 150 - 4 \times 1 = 146$
$T_2 = 150 - 4 \times 2 = 142$
$T_3 = 150 - 4 \times 3 = 138$ ✓✓

Hence the recursive rule is $T_{n+1} = T_n - 4$ with $T_1 = 146$ ✓ ✓

Calculator Free

4. [5 marks: 2, 3]

An arithmetic sequence is described by the rule $T_{n+1} = T_n + 6$ with $T_1 = -96$.

(a) Find the general rule of this sequence in the form $T_n = a + bn$, where a and b are constants and $n = 1, 2, 3, 4, 5, \ldots$.

Using the recursive rule:
the common difference = 6 ✓
General rule is $T_n = -102 + 6n$ ✓

(b) How many negative terms are there in this sequence?

General term rule is $T_n = -102 + 6n$
For negative terms: $-102 + 6n < 0$ ✓
$n < 17$ ✓
Hence, there are 16 negative terms. ✓

5. [9 marks: 4, 2, 3]

The sum of the first n terms of an arithmetic progression is given by $S_n = 3n^2 - 21n$. Find:

(a) the first three terms of the sequence.

When $n = 1$, $S_1 = 3 \times (1)^2 - 21 \times 1 = -18 \Rightarrow T_1 = -18$ ✓
When $n = 2$, $S_2 = 3 \times (2)^2 - 21 \times 2 = -30$ ✓
But, $T_2 = S_2 - S_1 = -30 - (-18) = -12$ ✓
When $n = 3$, $S_3 = 3 \times (3)^2 - 21 \times 3 = -36$
Hence, $T_3 = S_3 - S_2 = -36 - (-30) = -6$ ✓

(b) the recursive rule of the sequence.

First 3 terms are: –18, –12, –6.
Hence, recursive rule is $T_{n+1} = T_n + 6$ with $T_1 = -18$ ✓✓

(c) the sum of all terms between the 11th term and the 20th term inclusive.

Required Sum $= S_{20} - S_{10}$ ✓
$= 3(20^2) - 21(20) - [3(10^2) - 21(10)]$ ✓
$= 780 - 90 = 690$ ✓

Calculator Free

6. [5 marks]

The eighth term and twelfth term of an arithmetic sequence are 24 and 40 respectively. Find the recursive rule for the sequence.

> Difference between 12th term and 8th term = 40 – 24 = 16. ✓
>
> Twelfth term = Eighth term + 4 × common difference
> Hence, 4 × common difference = 16
> common difference = 4 ✓
>
> First Term = 24 – 7 × common difference
> = 24 – 7 × 4 = –4 ✓
>
> Recursive rule is: $T_{n+1} = T_n + 4$ with $T_1 = -4$ ✓✓

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7. [8 marks: 3, 4, 1]

The sixth term of an arithmetic sequence is double its fourth term. The first term of the sequence is 20 and the common difference is d.

(a) Show that $T_4 = 2 \times d$.

> Given $T_6 = 2 \times T_4$:
> $T_6 = T_4 + 2 \times d$ ✓
> $\Rightarrow 2 \times T_4 = T_4 + 2 \times d$ ✓
> $T_4 = 2 \times d$ ✓

(b) Hence, find the general rule for the sequence.

> $T_4 = 2 \times d$
> But $T_4 = 20 + 3 \times d$ ✓
> $2 \times d = 20 + 3 \times d$ ✓
> $d = -20$ ✓
> Hence, $T_n = 40 - 20n$ ✓

(c) Find the three consecutive terms of this sequence that sum to –60.

> The terms are 0, –20 and –40. ✓

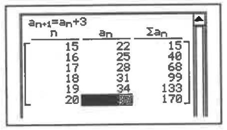
Calculator Assumed

8. [6 marks: 3, 1, 2]

An arithmetic sequence has first term –20 and common difference 3.

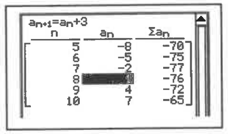
(a) Find the 20th term of the sequence and the sum of the first 20 terms.

> Use $T_{n+1} = T_n + 3$, $T_1 = -20$ ✓
> $T_{20} = 37$ ✓ $S_{20} = 170$ ✓
> OR
> $T_{20} = -20 + 19(3) = 37$ ✓✓
> $S_{20} = \frac{20}{2}(2 \times -20 + 19 \times 3) = 170$ ✓



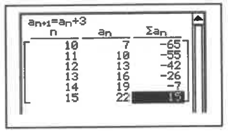
(b) Find the first positive term in the sequence.

> First positive term = 1 ✓
>
> OR
> $-23 + 3n > 0 \Rightarrow n > 7.7$
> Hence, $T_8 = 1$. ✓



(c) Find n so that the sum of the first n terms is positive for the first time.

> $S_{14} = -7$ and $S_{15} = 15$ ✓
> $n = 15$ ✓
> OR
> $\frac{n}{2}[2 \times -20 + (n-1) \times 3] > 0$ ✓
> Use "Solver": $n > 14.3 \Rightarrow n = 15$ ✓



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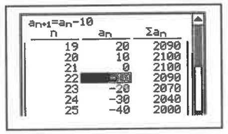
9. [5 marks: 3, 2]

An arithmetic sequence has first term 200 and common difference –10.

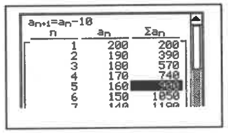
(a) Find which term is the first negative term in the sequence.

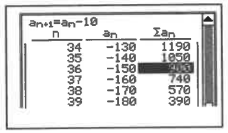
> Use $T_{n+1} = T_n - 10$, $T_1 = 200$ ✓
> $T_{21} = 0$, $T_{22} = -10$ ✓
> First negative term is T_{22} ✓
> OR
> $210 - 10n < 0 \Rightarrow n > 21$ ✓✓
> Hence, T_{22} ✓



(b) The sum of the first n terms is 900. Find n.

> $n = 5$ and 36 ✓✓
>
> OR
> $\frac{n}{2}[2 \times 200 + (n-1) \times 10] = 900$ ✓
> $n = 5$ and 36 ✓





Calculator Assumed

10. [9 marks: 2, 2, 3, 1, 1]

A special filter is installed to remove airborne particles from a dust-free room. The filter has to be changed every 10 weeks. In the first five weeks of operation, it removes 10 000 particles each week and thereafter its filtering capacity reduces by 500 particles each week.

(a) Write a recursive equation that describes the filtering capacity of the filter for the first five weeks.

$T_{n+1} = T_n$ ✓	where $T_1 = 10\,000$ ✓	$(1 \le n \le 5)$ *(optional)*

(b) Write a recursive equation that describes the filtering capacity of the filter from the sixth week to the tenth week inclusive.

$T_{n+1} = T_n - 500$ ✓	where $T_1 = 9\,500$ ✓	(T_1 is week 6.) *(optional)*

(c) Write in terms of k, an equation that describes the number of particles filtered in week k, if $6 \le k \le 10$.

$P = 10\,000 - 500(k-5)$ ✓✓
$= 12\,500 - 500k$ ✓

(d) Find the total number of particles filtered in the first 5 weeks.

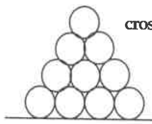
Total $= 10\,000 \times 5 = 50\,000$ ✓

(e) Find the total number of particles filtered by the end of the 7th week.

Total $= 50\,000 + 9\,500 + 9\,000 = 68\,500$ ✓

Calculator Assumed

11. [11 marks: 2, 2, 3, 4]

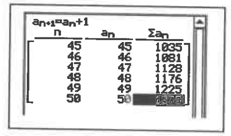


Joe runs a hardware store. He stores 25mm diameter cylindrical polyurethane pipes (each of length 10m) used for reticulation systems in the yard and stacks the pipes up in piles similar to the one shown in the accompanying diagram. Each pile has one pipe at the top of the pile.

(a) The bottom row of a pile of pipes has 50 pipes. How many pipes are there in this pile?

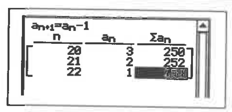
Since bottom pile has 50 pipes,
there are 50 rows. ✓
The number of pipes in the pile forms an arithmetic sequence with first term 1 and common difference 1. ✓

Use $T_{n+1} = T_n + 1$, $T_1 = 1$
Hence, $S_{50} = 1275$ ✓
OR $S_{50} = \frac{50}{2}(2 \times 1 + 49 \times 1) = 1275$ ✓



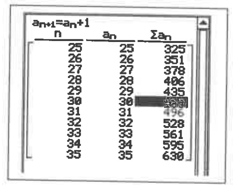
(b) Another pile has 18 pipes in its 5th row (row 1 is on the ground). How many pipes are there in this pile?

Since $T_5 = 18$, $T_1 = 18 + 4 = 22$ ✓
Hence, there are 22 rows.
Use $T_{n+1} = T_n - 1$, $T_1 = 22$
Therefore, $S_{22} = 253$ ✓
OR $S_{22} = \frac{22}{2}(2 \times 22 + 21 \times -1) = 253$ ✓✓



(c) There are 465 pipes in a pile. How many rows are there in this pile?

Use $T_{n+1} = T_n + 1$, $T_1 = 1$ ✓
Hence, need to $S_n = 465$ ✓
$n = 30$ ✓
OR
AP: $a = 1$, $d = 1$ ✓
$\frac{n}{2}[2 \times 1 + (n-1) \times 1] = 465$ ✓
$n = 30$ (reject -31) ✓



Calculator Assumed

11. (d) A new shipment of 100 pipes was delivered. How can the pipes be stacked so that a minimum number of piles are used?

The table below lists the number of pipes required for different number of rows in a pile.

No. of rows	No. of pipes
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55
11	66
12	78
13	91

Some systematic table or method ✓✓

From the table, 1 pile of 9 rows and 1 pile of 10 rows will give a combined total of 45 + 55 = 100 pipes. ✓✓

12. [6 marks: 3, 1, 2]

Brooke invests \$50 000 in an account that pays simple interest at a rate of 5% per year. The interest is paid at the end of each year and is not added to the principal. Let B(n) be the account balance at the end of n years.

(a) Find the recursive rule and general rule for the account balance after n years.

Simple interest per year = 50 000 × 0.05 = \$2 500.
The yearly account balances form a sequence:
50 000, (50 000 + 2500), (50 000 + 2×2500), ...
Hence: $B(n) = B(n-1) + 2500$, $B(0) = 50\,000$. ✓✓
General rule $B(n) = 50\,000 + 2500n$ ✓

(b) Find n when the account balance is \$75 000.

$50\,000 + 2500n = 75\,000 \Rightarrow n = 10$ ✓

(c) Find the minimum number of years required for the balance to exceed \$150 000.

$50\,000 + 2500n > 200\,000$
$n > 40$ ✓
Hence, at least 41 years. ✓

Calculator Assumed

13. [9 marks: 2, 2, 5]

X and Y are two campsites 200 km apart on the Bibbulman Track. Paige is the leader of a group of hikers that start off from Camp X towards Camp Y. This group of hikers walk 5 km on the first day, and thereafter increase their rate by an extra 0.6 km each day until a rate of 8 km a day is achieved. They then walk at this rate for the rest of the trip.

(a) On which day did Paige's group attain the maximum walking rate of 8 km per day?

The distances walked each day form the sequence:
5, 5.6, 6.2, 6.8, 7.2, 8.0 ✓
Hence, 8km is achieved on the 6th day. ✓

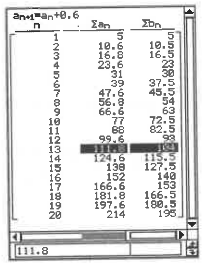
Jasmine is the leader of a second group of hikers that start off from Camp Y towards Camp X. This group of hikers walk 5 km on the first day, and thereafter increase their rate by an extra 0.5 km each day until a rate of 8 km a day is achieved. They then walk at this rate for the rest of the trip.

(b) On which day did Jasmine's group attain the maximum walking rate of 8 km per day?

The distances walked each day form the sequence:
5, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0 ✓
Hence, 8km is achieved on the 7th day. ✓

(c) If Paige's and Jasmine's group start off on the same day, when and where will the two groups meet along the Bibbulman Track? Show clearly how you obtained your answer.

For Paige's group: $a_{n+1} = a_n + 0.6$, $a_1 = 5$ ✓
For Jasmine's group: $b_{n+1} = b_n + 0.5$, $b_1 = 5$ ✓
Distance travelled by Paige's group after 12 days = 99.6 km
Distance travelled by Jamine's group after 12 days = 93 km
Distance covered by both groups after 12 days = 192.6 km ✓
Distance travelled by Paige's group after 13 days = 111.8 km
Distance travelled by Jamine's group after 13 days = 104 km
Distance covered by both groups after 13 days = 215.8 km ✓
Hence, the two groups will meet during the 13th day. ✓



OR
For Paige's group: AP: $a = 5$, $d = 0.6$ ✓
For Jasmine's group: AP: $a = 5$, $d = 0.5$ ✓

$\frac{n}{2}[2 \times 5 + (n-1) \times 0.6] + \frac{n}{2}[2 \times 5 + (n-1) \times 0.5] = 200$ ✓
$n = 12.3$ (reject $n = -29.5$). Hence, 13th day. ✓✓

Calculator Assumed

14. [9 marks: 3, 6]

A grandfather clock makes as many long chimes as the hour of the day, on the hour, every hour. For example, at 1 pm (or am) it makes one long chime, at 2 pm (or 2 am) it makes two long chimes, …. , at 12 midnight (or 12 noon) it makes twelve long chimes.

(a) How many long chimes would this clock make in a 24 hour day? Show clearly how you obtained your answer.

> $(1 + 2 + 3 + 4 + \ldots\ldots + 12) \times 2 = 156$ ✓ ✓ ✓

In addition, this clock makes 1 short chime to mark the passage of the first quarter of any hour, 2 short chimes to mark the passage of the first half-hour of any hour and 3 short chimes to mark the passage of the third quarter of any hour. For example:

- at 1 pm, it will make one long chime.
- at 1.15pm it will make 1 short chime, at 1.30 pm it will make 2 short chimes.
- at 1.45 pm it will make 3 short chimes and at 2 pm it will make two long chimes.

(b) How long after 12 noon, would it take for the clock to have made a total of 105 chimes (long and short)? Justify your answer.

Time	Long Chimes	Short Chimes	Total
Just after 12 noon and just before 1 pm	0	1 + 2 + 3 = 6	6
From 1 pm to just before 2 pm	1	6	7
From 2 pm to just before 3 pm	2	6	8
From 3 pm to just before 4 pm	3	6	9
From 4 pm to just before 5 pm	4	6	10

Clearly, the total number of chimes each hour after 12 noon forms an arithmetic sequence 6, 7, 8, 9, 10, …. . ✓✓

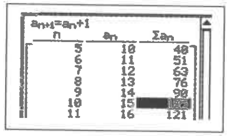
Hence, need to solve $(6 + 7 + 8 + 9 + 10 + \ldots\ldots) = 105$ ✓

Use $T_{n+1} = T_n + 1$, $T_1 = 6$ ✓

$S_{10} = 105$ ✓

Hence, 105 chimes is struck before the end of the 10th hour.

Hence, time taken = 10 hours – 15 minutes = 9 hours and 45 minutes ✓



25 Geometric Progressions

Calculator Free

1. [4 marks]

A geometric sequence has first term 5 and common ratio 2. State the recursive rule and general rule for this sequence.

> Recursive rule is $T_{n+1} = T_n \times 2$ with $T_1 = 5$ ✓ ✓
>
> General Rule is $T_n = 5 \times 2^{n-1}$ ✓ ✓

2. [5 marks: 3, 2]

The terms of a sequence are defined by $\frac{T_{n+1}}{T_n} - \frac{1}{2} = 0$ with $T_1 = 1024$

(a) Show that this sequence is a geometric sequence.

> Rewrite recursive rule as $\frac{T_{n+1}}{T_n} = \frac{1}{2}$ with $T_1 = 1024$ ✓
>
> Hence, ratio between any two consecutive terms is constant. ✓
>
> Hence, sequence is a geometric sequence. ✓

(b) How many terms greater than 32 are there in this sequence?

> Terms are 1024, 512, 256, 128, 64 , 32, 16 … ✓
>
> Hence, five terms. ✓

3. [4 marks]

A geometric sequence is described by the rule $T_n = 5 \times 3^n$, where $n = 1, 2, 3, \ldots$. Find the first three terms of the sequence. Hence, state the recursive rule for this sequence.

> $T_1 = 5 \times 3 = 15$
>
> $T_2 = 15 \times 3 = 45$
>
> $T_3 = 45 \times 3 = 135$ ✓✓
>
> Hence the recursive rule is $T_{n+1} = T_n \times 3$ with $T_1 = 15$ ✓ ✓

Calculator Free

4. [5 marks: 3, 2]

A sequence is described by the rule $T_{n+1} = T_n \times 5$ with $T_1 = 2$.

(a) Find the general rule of this sequence in the form $T_n = a \times b^n$, where a and b are constants and $n = 1, 2, 3, 4, 5, \ldots$.

> From the recursive rule, the common ratio = 5 ✓
> General rule is $T_n = 2 \times 5^{n-1}$ ✓
> $= 2 \times 5^{-1} \times 5^n = 0.4 \times 5^n$ ✓

(b) How many terms less than 1000 are there in this sequence?

> Terms are 2, 10, 50, 250, 1250, … . ✓
> Hence, four terms. ✓

---

5. [8 marks: 3, 2, 3]

The sum of the first n terms of a geometric progression is given by $S_n = 4^{n+1} - 4$.

(a) Find the first three terms of the sequence.

> When $n = 1$, $S_1 = 4^2 - 4 = 12 \Rightarrow T_1 = 12$ ✓
> When $n = 2$, $S_2 = 4^3 - 4 = 60$
> But, $T_2 = S_2 - S_1 = 60 - 12 = 48$ ✓
> Hence, common ratio is 4.
> Hence, $T_3 = 48 \times 4 = 192$ ✓

(b) Find the general rule of the sequence.

> General rule is $T_n = 12 \times 4^{n-1}$ ✓✓
> $= 3 \times 4^n$

(c) Find a mathematical expression for the sum of all terms between the 10th term and the 15th term inclusive.

> Required Sum $= S_{15} - S_9$ ✓
> $= (4^{16} - 4) - (4^{10} - 4)$ ✓
> $= 4^{16} - 4^{10}$ ✓

Calculator Free

6. [4 marks]

The general rule of a geometric sequence is given by $T_n = \frac{4}{10^n}$, where $n = 1, 2, 3,$

Find the sum to infinity of this sequence if it exists. Justify your answer.

> First term of sequence $a = \frac{4}{10} = \frac{2}{5}$. ✓
> Common ratio $r = \frac{1}{10}$. ✓
> Since $-1 < r < 1$, S_∞ exists. ✓
> $S_\infty = \frac{\frac{2}{5}}{1 - \frac{1}{10}} = \frac{4}{9}$ ✓

---

7. [4 marks: 2, 2]

(a) The sum of the first n terms of a geometric progression is given by $S_n = 5 \times 2.5^n - 5$. Determine the sum to infinity of this sequence if it exists.

> As $n \to \infty$, $2.5^n \to \infty$. ✓
> Hence, S_∞ does not exist. ✓

(b) The sum of the first n terms of a geometric progression is given by $S_n = 0.25(1 - 0.2^n)$. Determine the sum to infinity of this sequence if it exists.

> As $n \to \infty$, $0.2^n \to 0$. ✓
> Hence, $S_\infty = 0.25(1 - 0) = 0.25$. ✓

---

8. [3 marks]

The sum to infinity of a geometric sequence with first term 10 is 40. Find the recursive rule of this sequence.

> $S_\infty = \frac{a}{1-r} \Rightarrow 40 = \frac{10}{1-r}$ ✓
> $r = 0.75$ ✓
> $T_{n+1} = T_n \times 0.75$ with $T_1 = 10$. ✓

Calculator Assumed

9. [10 marks: 3, 2, 2, 3]

A sequence is described by the rule $T_n = 1500(1.04)^n$, where $n = 1, 2, 3, \ldots$

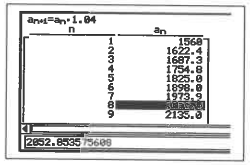
(a) State the recursive rule of this sequence.

When $n = 1$, $T_1 = 1500 \times (1.04)^1 = 1560$ ✓
Common ratio = 1.04 ✓

Hence recursive rule is $T_{n+1} = T_n \times 1.04$ where $T_1 = 1560$ ✓

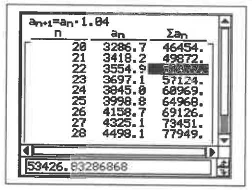
(b) Find the first term that exceeds 2 000.

Use $T_{n+1} = T_n \times (1.04)$ where $T_1 = 1560$
$T_7 = 1973.9$, $T_8 = 2052.85$ ✓
Hence, the 8th term. ✓
OR
$1500(1.04)^n > 2000$ ✓
$n > 7.3$
Hence, the 8th term. ✓



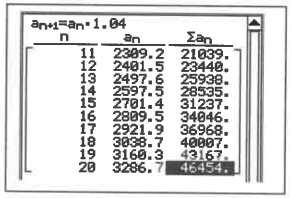
(c) Find the least value for n for which the sum of the first n terms is greater than 50 000.

Use $T_{n+1} = T_n \times (1.04)$ where $T_1 = 1560$.
$S_{21} = 49872$ and $S_{22} = 53427$ ✓
Hence, least value for n is 22. ✓
Or
$\frac{1560(1-1.04^n)}{1-1.04} > 50\,000$ ✓
$n > 21.04$
Hence, least value for n is 22. ✓



(d) Find the sum of the second set of ten terms.

Sum of 11th term through to 20th term
$= S_{20} - S_{10}$ ✓
$= 46\,453.80 - 18\,729.53$ ✓
$= 27\,242.3$ ✓
OR
Sum of 11th term through to 20th term
$= S_{20} - S_{10}$ ✓
$= \frac{1560(1-1.04^{20})}{1-1.04} - \frac{1560(1-1.04^{10})}{1-1.04}$ ✓
$= 46\,453.80 - 18\,729.53 = 27\,242.3$ ✓



Calculator Assumed

10. [6 marks: 3, 3]

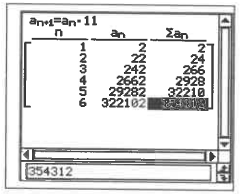
The fourth term and the ninth term of a geometric sequence are respectively 2662 and 428 717 762.

(a) Find the common ratio of this sequence.

$T_9 = T_4 \times r^5$ ✓
$r^5 = \frac{428\,717\,762}{2662} = 161\,051$ ✓
$r = 11$ ✓

(b) Find the sum of the first six terms of this sequence.

First term $= T_4 \div r^3$
$= \frac{2662}{11^3} = 2$ ✓
Use $T_{n+1} = T_n \times 11$ where $T_1 = 2$ ✓
$S_6 = 354\,312$ ✓
Or
$S_6 = \frac{2(1-11^6)}{1-11} = 354\,312$ ✓✓



11. [5 marks: 2, 3]

In its first year of operation, a recycling plant processed 2 000 tonnes of recyclable waste **per month**. In the second year it processed 3000 tonnes per month and in the third year it processed 4500 tonnes per month. Successive amounts form a geometric sequence until year five when it reached its processing capacity.

(a) What was the maximum processing capacity of the recycling plant per month?

Capacities are: 2000, 3000, 4500, 6750, 10 125 ✓
Hence, maximum capacity = 10 125 tonnes/month ✓

(b) Determine the total amount of waste processed for the first **five years**.

Total = (2000 + 3000 + 4500 + 6750 + 10 125) × 12 ✓✓
= 316 500 tonnes ✓

Calculator Assumed

12. [10 marks 2, 2, 2, 2, 2]

Peter hopes to save enough money to buy himself a new computer. In his plan, which covers a period of a fortnight, he saves 10 cents on the first day, 20 cents on the second day, 40 cents on the third day etc. , each time doubling the amount he saved the previous day.

(a) How much would Peter have to save on the 10th day?

> Daily savings: 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, 6.4, 12.8, 25.6, 51.2, .. ✓
> or use $T_{n+1} = T_n \times 2$ where $T_1 = 0.1$ or $T_n = 0.1 \times 2^{n-1}$
> Hence, savings on 10th day, T_{10} = \$51.20 ✓

(b) How much would Peter have saved by the end of the 10th day?

> Use $T_{n+1} = T_n \times 2$ where $T_1 = 0.1$ ✓
> Or $S_{10} = \dfrac{0.1(1-2^{10})}{1-2}$ = \$102.30 ✓

(c) On which day would he have to save \$3.20?

> Sequence is 0.1, 0.2, 0.4, 0.8, 1.6, 3.2 ✓
> or use $T_{n+1} = T_n \times 2$ where $T_1 = 0.1$ or use $T_n = 0.1 \times 2^{n-1}$
> Hence , on the sixth day. ✓

(d) How many days would he take to save a total of at least \$20.00?

> Use $T_{n+1} = T_n \times 2$ where $T_1 = 0.1$ Or $S_n = \dfrac{0.1(1-2^n)}{1-2}$
> S_7 = \$12.70 and S_8 = \$25.50 ✓
> Hence, 8 days. ✓

(e) Comment on whether his savings plan is realistic.

> Not realistic. ✓
> He would have to save \$819.20 on the 14th day
> which is more than an average worker's weekly wage! ✓

---

13. [7 marks: 2, 1, 2, 2]

A toad hops 200 m along an outback highway on the first night. The distance hopped each night is 1% less than the distance hopped the previous night.

(a) How far does the toad hop during the 6th night?

> Use $T_{n+1} = T_n \times 0.99$ where $T_1 = 200$ ✓
> Distance hopped, T_6 = 190.198 m ✓ (or 200×0.99^5)

Calculator Assumed

13. (b) How far would the toad have hopped by the end of the 6th night?

> Use $T_{n+1} = T_n \times 0.99$ where $T_1 = 200$
> Distance hopped = S_6 = 1170.397 m ✓

(c) Find the least number of nights required for the toad to have hopped a total distance of at least 10 km. Justify your answer.

> Use $T_{n+1} = T_n \times 0.99$ where $T_1 = 200$
> $S_{68} = 9902$, $S_{69} = 10\,003$ ✓
> Hence, least number of nights is 69. ✓

(d) Will the toad be able to reach a roadhouse 25 km from where it started hopping? Justify your answer.

> $S_\infty = \dfrac{200}{1-0.99}$ = 20 km < 25 km ✓
> Hence, no! ✓

---

14. [10 marks: 2, 3, 3, 2]

An observation balloon is released from a height of 100 metres and allowed to float vertically upwards. The height increase in the first minute is 10 metres. Thereafter, the height increase during each subsequent minute is 70% of the height increase during the previous minute. Ignore air and wind resistance.

(a) Find the height increase during the 8th minute.

> Use $T_{n+1} = T_n \times 0.7$ where $T_1 = 10$ ✓
> Hence, height increase T_8 = 0.82 m ✓ (or 10×0.7^7)

(b) Find the height at the end of the 8th minute.

> Height = 100 + S_8 (for $T_{n+1} = T_n \times 0.7$ where $T_1 = 10$) ✓✓
> = 100 + 31.41 = 131.41 m ✓

(c) Find when the height of the balloon first exceeds 120 metres.

> Height increase = 120 – 100 = 20 m ✓
> $10 + 10 \times 0.7 + 10 \times 0.7^2$ = 21.9 m ✓
> Hence, during the 3rd minute. ✓

(d) Determine with reasons if the balloon will ever reach a height of 135 m.

> $S_\infty = \dfrac{10}{1-0.7}$ = 33.33 m
> Hence, maximum possible height = 100 + 33.33 = 133.33 < 135 m. ✓
> Therefore, no! ✓

Calculator Assumed

15. [13 marks: 2, 3, 2, 3, 3]

A robot submarine was released from a depth of 100 metres. During the first minute it descends a vertical distance of 20 metres. During the second minute it descends (vertically) a further 16 metres. The vertical descent each subsequent minute is 80% of the descent of the previous minute.

(a) Find the vertical descent during the 5th minute.

> Descents are 20, 20×0.8, 20×0.8^2, 20×0.8^3, 20×0.8^4
> Hence, descent during the 5th minute, $T_5 = 20 \times 0.8^4 = 8.192$ m ✓✓

(b) Find the depth of the robot submarine at the end of the 5th minute.

> Depth at end of 5th minute
> $= 100 + S_5$ (for $T_{n+1} = T_n \times 0.80$ where $T_1 = 20$) ✓✓
> $= 100 + 67.232$
> $= 167.232$ m. ✓

(c) During which minute did the robot submarine descend by 2.147 m?

> Use $T_{n+1} = T_n \times 0.80$ where $T_1 = 20$
> $T_{11} = 2.147$ ✓
> Hence, during the 11th minute. ✓

(d) Find when the robot submarine reached a depth of 180 metres.

> Distance descended = 180 – 100 = 80 ✓
> Use $T_{n+1} = T_n \times 0.80$ where $T_1 = 20$
> $S_7 = 79.03$ m $S_8 = 83.2$ m ✓
> Hence, during the 8th minute. ✓

(e) The seabed is 250 metres below sea level. Will the robot submarine ever reach the sea bed? Justify your answer.

> $S_\infty = \dfrac{20}{1 - 0.8} = 100$ m. ✓
> Hence, maximum depth = 100 + 100
> = 200 (which is less than 250 m) ✓
> Hence, the submarine will never reach the sea-bed. ✓

Calculator Assumed

16. [10 marks: 2, 2, 3, 3]

A rubber ball is dropped from a height of 200 cm. Each time it hits the ground it will bounce vertically upwards to a height that is 80% of the height it reached in the previous bounce. It bounced to a height of 150 cm after it hit the ground the first time.

(a) Find the height reached by the ball after it hits the ground for the 3rd time.

> Heights reached: 150, 150×0.8, 150×0.8^2 ✓
> Hence, height reached = $150 \times 0.8^2 = 96$ cm. ✓

(b) After how many times would the ball have to hit the ground before it first rebounds to a height less than 50 cm.

> Use $T_{n+1} = T_n \times 0.8$ where $T_1 = 150$
> $T_5 = 61.44$ $T_6 = 49.15$ ✓
> Hence, need to hit the ground 6 times. ✓

(c) Find the total distance travelled by the ball just before it hits the ground for the 5th time.

> Total distance = $200 + 2 \times S_4$ (for $T_{n+1} = T_n \times 0.8$ where $T_1 = 150$) ✓✓
> $= 200 + 2 \times 442.8$
> $= 1\,085.6$ cm ✓

(d) Find the total distance travelled by the ball before it comes to rest on the ground.

> Total distance = 200 ✓
> $+ 2 \times S_\infty$ of GP ($a = 150$, $r = 0.8$) ✓
> = 1700 cm ✓

17. [7 marks: 2, 3, 2]

An investment account pays 15% interest compounded annually over a 15 year period. Brad invests $100 000 in this account for 15 years. No new money was added to and no withdrawals were made from the investment account.

(a) Calculate the value of the investment account after 10 years.

> Let T_n : Value after n years.
> $T_n = 100000 \times 1.15^{10}$ ✓
> $= \$404\,555.77$ ✓
> Or use $T_{n+1} = T_n \times 1.15$ where $T_1 = 100\,000 \times 1.15$ ✓
> $T_{10} = \$404\,555.77$ ✓

Calculator Assumed

17. (b) Calculate the increase in value of the account during the 10th year.

> Value after 10 years $T_{10} = 100000 \times 1.15^{10} = \$404\,555.77$
> Value after 9 years $T_9 = 100000 \times 1.15^9 = \$351\,787.63$
> Hence, increase during the 10th year = 404555.77 – 351787.63 ✓✓
> = \$52 768.14 ✓

(c) Calculate the minimum number of years required for the initial amount invested to double.

> $100\,000 \times 1.15^n = 200\,000$ ✓
> $n = 4.96$
> Hence need 5 years. ✓

---

18. [10 marks: 2, 3, 3, 2]

\$1 000 000 is invested in an account that pays interest at a rate of 5% per annum compounded annually. Let B(n) be the account balance at the end of n years.

(a) Find the general rule for the account balance at the end of n years.

> General rule $B(n) = 1\,000\,000 \times 1.05^n$ ✓✓

(b) Find the growth in the account balance in the first 10 years.
Hence, find the average percentage growth rate in the first 10 years.

> $B(10) = 1\,000\,000 \times 1.05^{10} = \$1\,628\,894.627$ ✓
> Growth = \$628 894.627 ✓
> Hence, average % growth rate = $\frac{628\,894.627}{1\,000\,000} \times \frac{1}{10} \times 100$
> = 6.29% ✓

(c) Calculate the average percentage growth rate in the first 20 years.

> $B(20) = 1\,000\,000 \times 1.05^{20} = \$2\,653\,297.705$ ✓
> Growth = \$1 653 297.705 ✓
> Hence, average % growth rate = $\frac{1\,653\,297.705}{1\,000\,000} \times \frac{1}{20} \times 100$
> = 8.27% ✓

(d) Give an explanation for the different answers in parts (b) and (c).

> Due to compounding effects, the annual growth increases from year to year. ✓
> Hence, there is proportionally a larger increase over 20 years than 10 years and therefore a higher growth rate. ✓

Calculator Assumed

19. [9 marks: 3, 3, 3]

To fight an infection, Steele has to take a course of medication which consists of 10 tablets to be taken over 10 days. One tablet is to be taken at the same time each day. Each tablet contains 50 mg of a particular drug. At the end of each 24 hour period, only 20% of the drug remains in the body. The table below models the amount of the drug in the body for a period of 5 days.

Day	Amount of drug in the body (mg)	
	Just before tablet is taken	Just after the tablet is taken
1	0	50
2	50×0.2	$50 + 50 \times 0.2$
3	$(50 \times 0.2) + (50 \times 0.2^2)$	$50 + (50 \times 0.2) + (50 \times 0.2^2)$
4	$(50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3)$	$50 + (50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3)$
5	$(50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3) + (50 \times 0.2^4)$	$50 + (50 \times 0.2) + (50 \times 0.2^2) + (50 \times 0.2^3) + (50 \times 0.2^4)$

(a) Find the amount of drug left in the body just before the 6th tablet is taken.

> Let $T_{n+1} = T_n \times 0.2$ where $T_1 = 50 \times 0.2$ ✓✓
> Amount left = S_5
> = 12.496 mg ✓

(b) Find the amount of drug in the body just after the 10th tablet is taken.

> Let $T_{n+1} = T_n \times 0.2$ where $T_1 = 50$ ✓
> Amount left = S_{10}
> = 62.5 mg ✓

(c) Find the amount of drug left in the body one week after the last tablet was taken. Comment on your answer.

> Amount left just after the 10th tablet was taken = 62.5 mg ✓
> Hence, amount left after 7 days = 62.5×0.2^7 ✓
> = 0.0008 mg ✓
> Hence, there is virtually no trace of the drug left.

Calculator Assumed

20. [14 marks: 1, 4, 4, 5]

A group of engineers plan to bore a horizontal tunnel underneath a city to accommodate a railway line. The tunnel needs to be 3 km long. Assume that the tunnel is in a straight horizontal line. A tunnel boring machine is assembled and used to bore the tunnel. On the first ten days, the machine bores a distance of 100 m each day. Between the 11th and 20th day inclusive, because of the fragile nature of the environment the distance bored each day is 90% of the distance bored in the previous day (the length bored is 90 m on the 11th day). From and on the 21st day onwards, the machine bores an extra 10 m more than what was bored in the previous day.

(a) Find the total distance bored in the first 10 days.

> Total distance bored = 100 × 10 = 1000 m. ✓

(b) How much *less distance* is bored on the 20th day compared to the 19th day?

> For day 11 onwards inclusive, use $T_{n+1} = T_n \times 0.9$ where $T_1 = 100 \times 0.9$ ✓
> Distance bored on the 19th day, $T_{19} = 38.74$ ✓ (or 100×0.9^9)
> Distance bored on the 20th day, $T_{20} = 34.87$ ✓ (or 100×0.9^{10})
> Hence, (38.74 – 34.87) = 3.87 m less. ✓

(c) Find the total distance bored in the first 20 days (2 decimal places). Show full working out.

> Total distance bored in first 10 days = 1000 m. ✓
>
> Total distance bored between 11th day and 20th day inclusive
> = S_{10} (for $T_{n+1} = T_n \times 0.9$ where $T_1 = 100 \times 0.9$) ✓
> = 586.1894 m ✓
>
> Hence, total distance bored = 1000 + 586.1894 = 1586.19 m. ✓

(d) How many days are required to complete boring the entire 3 km tunnel? Justify your answer.

> After the first 20 days, distance remaining = 3000 – 1586.19
> = 1413.81 m ✓
> Distance bored on day 20 = 34.87,
> Distance bored on day 21 = 34.87 + 10 = 44.87
> Distance bored on day 22 = 44.87 + 10 = 54.87
> Hence, need to solve (34.87 + 44.87 + 54.87 + ...) = 1413.81 ✓
> Use $T_{n+1} = T_n + 10$ where $T_1 = 34.87$ ✓
> $S_{14} = 1398.18$
> $S_{15} = 1573.05$ ✓
> Hence, on the 15th day after 19 days, the tunnel would be completed.
> Hence, number of days required = 19 + 15 = 34. ✓

26 Exponential Functions II

Calculator Assumed

1. [6 marks: 1, 2, 3]

Q, the number of organisms (*in hundreds*) in a laboratory culture is related to time t (days) by the formula $Q = 16 \times (1.075)^t$.

(a) How many organisms were there at the start?

> $Q(0) = 16 \times 1.075^0 = 16$
> Hence, no. of organisms = 16 × 100 = 1600 ✓

(b) Find the number of organisms after one week.

> $Q(7) = 16 \times 1.075^7 = 26.5447$ ✓
> Hence, no. of organisms = 26.5447 × 100 = 2654 ✓

(c) How long will it take for the population to reach 2 000? *Show clearly the method you used.* Give your answer to the nearest day.

> When population = 2000, Q = 20 ✓
> $20 = 16 \times 1.075^t$ ✓
> Use "solver": $t = 3.09$
> Hence, 4 days. ✓

---

2. [7 marks: 1, 3, 3]

The amount of radioactive substance at time t years is given by $A = 150\,(0.8)^{t+1}$ g.

(a) How much radioactive substance was there at the start?

> $A(0) = 150 \times 0.8^1 = 120$ g ✓

(b) Find the amount of radioactive substance that has decayed after 10 years.

> $A(10) = 150 \times 0.8^{11} = 12.8849$ ✓
> Hence, amount decayed = 120 – 12.8849 ✓
> = 107.115 g ✓

(c) How long will it take for half the original amount to decay? *Show clearly the method you used.* Give your answer to the nearest year.

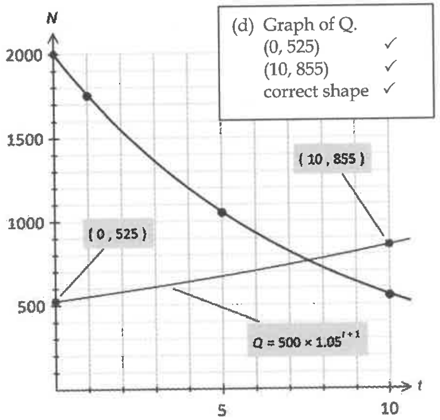
> $150 \times 0.8^{t+1} = 60$ ✓
> Use "solver": $t = 3.106$ ✓
> Hence, after 3.106 years (4 years) ✓

Calculator Assumed

3. [10 marks: 2, 1, 2, 5]

At the commencement of the use of a desalination plant, the number of dolphins at a cove near the desalination plant was estimated to be 2000. The number of dolphins, N, was monitored for several years and is displayed in the table below.

Years after, t	Number of dolphins, N
0	2000
1	1750
5	1050
10	550



(d) Graph of Q. ✓
(0, 525) ✓
(10, 855) ✓
correct shape ✓

The accompanying graph plots the points from the given table onto a set of axes.

The relationship between N and t is of the form $N = a(k^t)$.

(a) Use an appropriate method to find the values of a and k.
Give the value of a to the nearest 100 and the value of k to 2 decimal places.

> Using an appropriate applet/programme/routine:
> $N = 2000 \times 0.88^t \Rightarrow a = 2000,\ k = 0.88$ ✓✓

(b) Predict the population after 20 years.

> When $t = 20$, $N(20) = 2000 \times 0.88^{20} = 155$ ✓

(c) How many years will it take for the dolphin population to reach 100?

> When $N = 100$, $100 = 2000 \times 0.88^t$ ✓
> Use "solver": $t = 23.4$ years ✓

(d) The population of another marine animal in the area was also studied over the same time period and its population, Q, is given as $Q = 500\,(1.05)^{t+1}$. Draw the graph of Q onto the diagram given and use the graphs drawn to determine when the two populations are equal?

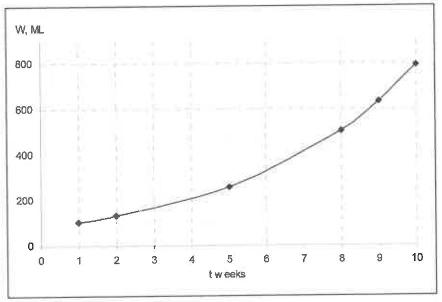
> From the graphs drawn, the two populations are equal between $t = 7$ and $t = 8$. ✓
> Hence, during the 8th year ✓

Calculator Assumed

4. [9 marks: 3, 2, 3, 1]

The amount of water, W MegaLitres, in a newly constructed dam at time t weeks is shown in the graph below. Three models were suggested for this data:

$$W = 80 \times 1.25^t$$
$$W = 100 \times 1.26^t$$
$$W = 50 \times 1.26^t$$



(a) Which of these three models best represent the data given?
Justify your answer.

> From graph, when $t = 1$, $W \approx 100$ and when $t = 10$, $W \approx 800$ ✓
> Eqn I. $W(1) = 100$ $W(10) = 745$
> Eqn II. $W(1) = 126$ $W(10) = 1009$
> Eqn III. $W(1) = 63$ $W(10) = 504$ ✓
> Hence, Eqn. I fits the curve best. $W = 80 \times 1.25^t$ ✓

(b) Use your chosen model to:
(i) estimate the amount of water in the dam after 20 weeks.

> $W(20) = 80 \times 1.25^{20}$ ✓
> $= 6938.89$ ML ✓

(ii) find when the amount of water in the dam will first exceed 3 000 ML.

> $3000 = 80 \times 1.25^t$ ✓
> Use "solver" $t = 16.2$ ✓
> Hence, during the 17th week. ✓

(c) What is the most important assumption underlying the model you chose in part (b)?

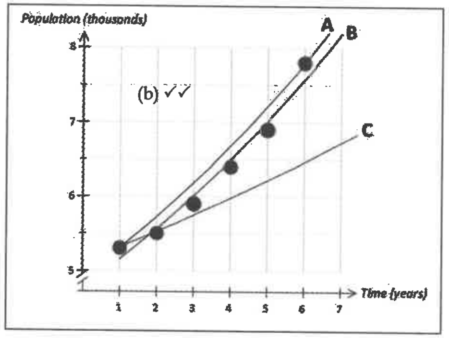
> The rate of increase remains constant throughout. ✓

Calculator Assumed

5. [12 marks: 3, 2, 2, 2, 3]

The fox population (in thousands) in a forest in the South West of the state is displayed in the table below:

After t years	Population, P (thousands)
1	5.3
2	5.5
3	5.9
4	6.4
5	6.9
6	7.8



Three models are proposed to describe the fox population as tabulated.

$P = 4.91 \times 1.08^t$ $\quad P = 5.11 \times 1.04^t$ $\quad P = 4.77 \times 1.08^t$

The graphs of the proposed models are drawn above.

(a) Match the curves drawn with the models given.

A. $P = 4.91 \times 1.08^t$ ✓ B. $P = 4.77 \times 1.08^t$ ✓ C. $P = 5.11 \times 1.04^t$ ✓

(b) Plot the values of P as displayed in the table onto the given graph.

(c) Which of the three given models best describe the tabulated data. Explain clearly how you arrived at your answer.

Best model is graph B ✓ as the plotted points are closest to this curve. ✓

(d) Use your chosen model to find:

(i) the population after 10 years.

$P(10) = 4.77 \times 1.08^{10} = 10.298$ ✓
Hence, population = 10.298 × 1000 = 10 298 ✓

(ii) when the population reaches 9 000 (*Give answer to the nearest month*).

When population = 9000, $P = 9$ ✓
Use "solver" to solve $4.77 \times 1.08^{10} = 9$ ✓
$\Rightarrow t = 8.2494$ years
= 8 years 3 months ✓

27 Differentiation

Calculator Free

1. [10 marks: 1, 1, 2, 3, 3]

Differentiate with respect to x:

(a) $3x^2 - 4x + 10$

Derivative = $6x - 4$ ✓

(b) $\dfrac{2}{3x^2} - x - 1$

Derivative = $\dfrac{-4}{3x^3} - 1$ ✓

(c) $\dfrac{1}{2\sqrt{x}} + \dfrac{5\sqrt{x}}{2}$

Derivative = $\dfrac{-1}{4x^{3/2}} + \dfrac{5}{4\sqrt{x}}$ ✓✓

(d) $\dfrac{3x^4 + x^3}{4x}$

$\dfrac{3x^4+x^3}{4x} = \dfrac{3x^4}{4x} + \dfrac{x^3}{4x} = \dfrac{3x^3}{4} + \dfrac{x^2}{4}$ ✓

Derivative = $\dfrac{9x^2}{4} + \dfrac{x}{2}$ ✓✓

(e) $(2x+1)^3$

$(2x+1)^3 = (2x)^3 + 3(2x)^2 + 3(2x) + 1$
$= 8x^3 + 12x^2 + 6x + 1$ ✓✓
Derivative = $24x^2 + 24x + 6$ ✓

2. [4 marks: 2, 2]

Given $f(x) = x^3 + x^2 - x + 4$, find:

(a) $f'(1)$

$f'(x) = 3x^2 + 2x - 1$ ✓
$f'(1) = 3 + 2 - 1 = 4$ ✓

Calculator Free

2. (b) x if $f'(x) = 0$.

$f'(x) = 3x^2 + 2x - 1 = 0$
$(3x - 1)(x + 1) = 0$ ✓
$x = -1, 1/3$ ✓

3. [2 marks]

Find the gradient of the curve $y = x^2 + 2\sqrt{x} + 1$ at the point where $x = 1$.

$y' = 2x + \frac{1}{\sqrt{x}}$ ✓
$y'(1) = 3$ ✓

4. [5 marks]

Find the equation of the tangent to the curve $y = \frac{x^2 - x^3}{x^4}$ at the point where $x = -1$.

$y = \frac{x^2 - x^3}{x^4} = x^{-2} - x^{-1}$ ✓
$y' = \frac{-2}{x^3} + \frac{1}{x^2}$ ✓
$y'(-1) = 3$ ✓
When $x = -1$, $y = 2$.
Equation of tangent is $y - 2 = 3(x + 1)$
$y = 3x + 5$ ✓✓

5. [5 marks]

Find the coordinates of the point(s) on the curve $y = \frac{1}{x} + x$ with a gradient of 0.

$y' = \frac{-1}{x^2} + 1$ ✓
Gradient = 0 ⇒ $\frac{-1}{x^2} + 1 = 0$ ✓
$x^2 = 1$
⇒ $x = -1$ or 1. ✓
Hence, (–1, –2) and (1, 2). ✓✓

Calculator Free

6. [5 marks]

The curve $y = ax^3 + bx^2 + 4x + 1$ has a gradient of 2 at the point (–1, –4). Find a and b.

When $x = -1$, $y = -4$.
Subst. into equation: $-4 = -a + b - 4 + 1$ ⇒ $-a + b = -1$ (I) ✓
Gradient function $\frac{dy}{dx} = 3ax^2 + 2bx + 4$ ✓
When $x = -1$, $\frac{dy}{dx} = 2$.
Hence, $2 = 3a - 2b + 4$ ⇒ $3a - 2b = -2$ (II) ✓
Solve I and II simultaneously; $a = -4$ and $b = -5$ ✓✓

7. [6 marks]

Given that $y = ax^3 + bx^2 + 2$ has a tangent with equation $y = -4x + 5$ at the point where $x = 1$, find a, and b.

$\frac{dy}{dx} = 3ax^2 + 2bx$ ✓
When $x = 1$, $\frac{dy}{dx} = -4$:
$3a + 2b = -4$ (1) ✓
Also, on the tangent:
when $x = 1$, $y = -4 + 5 = 1$. ✓
Hence, for the curve,
$a + b + 2 = 1$ ⇒ $a + b = -1$ (2) ✓
Solve (1) and (2) simultaneously:
$a = -2, b = 1$ ✓✓

Calculator Free

8. [6 marks]

A curve has equation $y = \frac{x^3}{3} + \frac{x^2}{2} - 4x + 1$. The points A and B lie on this curve and the tangents to the curve at A and B are parallel to the line $2x - y = 5$. Find the coordinates of the points A and B.

The line $2x - y = 5$ has gradient 2. ✓

$y = \frac{x^3}{3} + \frac{x^2}{2} - 4x + 1$

$\frac{dy}{dx} = x^2 + x - 4$ ✓

Hence, when $\frac{dy}{dx} = 2 \Rightarrow x^2 + x - 4 = 2$ ✓

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$\Rightarrow x = -3, 2$ ✓

Hence, A$(-3, \frac{17}{2})$ and B$(2, -\frac{7}{3})$. ✓✓

9. [7 marks]

The tangent to the curve $y = x^3(x+2)$ at the points where $x = 1$ and $x = -1$ meet at the point Q. Find the coordinates of the point Q.

Rewrite equation as $y = x^4 + 2x^3$

Gradient function $\frac{dy}{dx} = 4x^3 + 6x^2$ ✓

When $x = 1, y = 3$ and $\frac{dy}{dx} = 4 + 6 = 10$ ✓

When $x = -1, y = -1$ and $\frac{dy}{dx} = -4 + 6 = 2$ ✓

Equation of tangent at $x = 1$ is $y = 10x + c$ with $x = 1, y = 3$

$3 = 10 + c \Rightarrow c = -7$

Hence, equation is $y = 10x - 7$. ✓

Equation of tangent at $x = -1$ is $y = 2x + c$ with $x = -1, y = -1$

$-1 = -10 + c \Rightarrow c = 1$

Hence, equation is $y = 2x + 1$. ✓

At point of intersection: $y = 10x - 7$

$y = 2x + 1$ ✓

Solve simultaneously: $x = 1, y = 3$.

Hence, point of intersection of tangents is $(1, 3)$. ✓

Calculator Free

10. [9 marks: 6, 3]

Find the equation of the tangent(s) to the curve $y = \frac{x^3}{3} - x^2 - \frac{1}{3}$ that are:

(a) parallel to the line $x + y = 6$.

Gradient of line $= -1$ ✓

Gradient function of curve $\frac{dy}{dx} = x^2 - 2x$ ✓

For tangent to be parallel to the given line, $\frac{dy}{dx} = -1$

$x^2 - 2x = -1 \Rightarrow x^2 - 2x + 1 = 0$ ✓

$(x-1)(x-1) = 0$ $x = 1$ ✓

When $x = 1, y = \frac{1}{3} - (1)^2 - \frac{1}{3} = -1$ ✓

Equation of tangent at $x = 1$ is $y = -x + c$ with $x = 1, y = -1$

$-1 = -1 + c \Rightarrow c = 0$

Hence, equation is $y = -x$ ✓

(b) perpendicular to the line $x - y = 1$.

Gradient of given line $= 1$ ✓

For tangent to be perpendicular to this line, gradient of required line $= -1$. ✓

Hence, equation of tangent is $y = -x$. ✓

11. [8 marks]

A curve has equation $y = (x-2)(2x^2 - 5x + 2)$. The points A and B lie on this curve. The tangents to the curve at A and B are parallel to the line $12x - y = 5$. Find the coordinates of the points A and B.

$\frac{dy}{dx} = (2x^2 - 5x + 2) + (x-2)(4x-5)$ ✓

$= (2x^2 - 5x + 2) + (4x^2 - 13x + 10)$

$= 6x^2 - 18x + 12$ ✓

Given line has gradient 12. ✓

Tangents are parallel to line with gradient 12.

Hence, $6x^2 - 18x + 12 = 12$ ✓

$x^2 - 3x + 2 = 2$

$x(x-3) = 0$

$x = 0, 3$ ✓✓

Hence, points are $(0, -4)$ and $(3, 5)$ ✓✓

Calculator Free

12. [3 marks]

Use first principles to determine the derivative of $y = 5x^2$.

Let $f(x) = 5x^2 = 5 \times x^2$

$$\frac{dy}{dx} = 5 \times \lim_{h \to 0}\left[\frac{(x+h)^2 - x^2}{h}\right] \quad ✓$$

$$= 5 \times \lim_{h \to 0}\left[\frac{x^2 + 2xh + h^2 - x^2}{h}\right] \quad ✓$$

$$= 5 \times 2x = 10x \quad ✓$$

13. [4 marks]

Use first principles to determine the derivative of $y = \dfrac{1}{x^2}$.

Let $f(x) = \dfrac{1}{x^2}$.

$$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right] \quad ✓$$

$$= \lim_{h \to 0}\left[\frac{\left(\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right)}{h}\right] \quad ✓$$

$$= \lim_{h \to 0}\left[\frac{-2xh - h^2}{x^2(x+h)^2 h}\right]$$

$$= \lim_{h \to 0}\left[\frac{-2x - h}{x^2(x+h)^2}\right] \quad ✓$$

$$= \frac{-2x}{x^4}$$

$$= \frac{-2}{x^3} \quad ✓$$

Calculator Free

14. [2 marks]

Use an appropriate derivative to evaluate $\lim_{h \to 0}\left[\dfrac{\sqrt{2(x+h)} - \sqrt{2x}}{h}\right]$.

$$\lim_{h \to 0}\left[\frac{\sqrt{2(x+h)} - \sqrt{2x}}{h}\right] = \frac{d}{dx}\left(\sqrt{2x}\right) = \frac{1}{\sqrt{2x}} \quad ✓✓$$

15. [4 marks]

Use an appropriate derivative to evaluate $\lim_{h \to 0}\left[\dfrac{\left(1+\sqrt{5+h}\right)^2 - \left(1+\sqrt{5}\right)^2}{h}\right]$

giving your answer in exact form. Show clearly how you obtained your answer.

$$\lim_{h \to 0}\left[\frac{\left(1+\sqrt{5+h}\right)^2 - \left(1+\sqrt{5}\right)^2}{h}\right] = \frac{d}{dx}\left(1+\sqrt{x}\right)^2\Big|_{x=5} \quad ✓$$

$$= 2\left(1+\sqrt{x}\right) \times \frac{1}{2\sqrt{x}}\Big|_{x=5} \quad ✓$$

$$= \left(1+\sqrt{5}\right) \times \frac{1}{\sqrt{5}} \quad ✓$$

$$= \frac{1+\sqrt{5}}{\sqrt{5}} \quad ✓$$

$$= 1 + \frac{\sqrt{5}}{5}$$

28 Derivatives & Graphs

Calculator Free

1. [6 marks: 2, 2, 2]

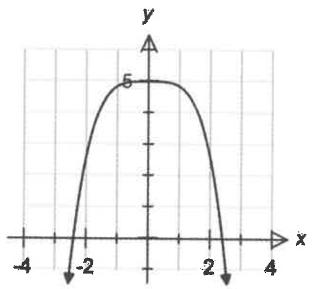
Use the sketch of $y = f(x)$ to determine the gradient of the curve at the points corresponding to the indicated values of x

(a) (i) $x = -2$

Gradient ≈ 5 ✓

(ii) $x = 1$

Gradient ≈ −0.6 ✓

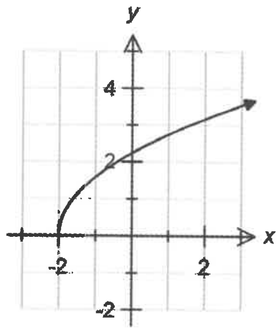


(b) (i) $x = -2$

Gradient → ∞ ✓

(ii) $x = 0$

Gradient ≈ 0.6 ✓

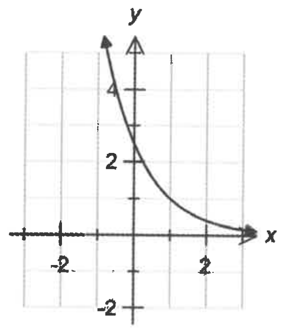


(c) (i) $x = 0$

Gradient ≈ −2.3 ✓

(ii) $x = 2$

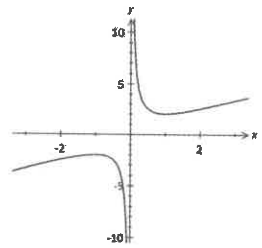
Gradient ≈ −0.4 ✓



Calculator Free

2. [4 marks: 2, 2]

The graph of $y = f(x)$ is given in the accompanying diagram.



(a) Find the x-coordinate of the point(s) where the gradient of the curve is 0.

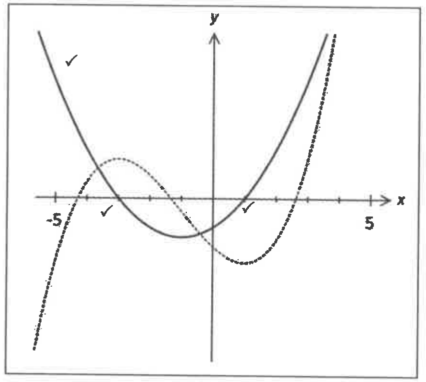
$x = -1$ and 1 ✓✓

(b) For what values of x is the gradient of the curve negative?

$-1 < x < 0$ and $0 < x < 1$ ✓✓

3. [5 marks: 2, 3]

The graph of $y = f(x)$ is given below.



(a) For what values of x is the gradient negative?

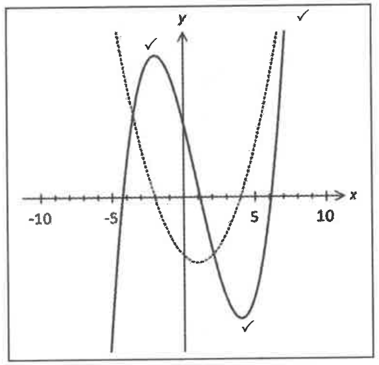
$-3 < x < 1$ ✓✓

(b) Sketch on the same axes, a possible graph of $y = f'(x)$.

Calculator Free

4. [6 marks: 2, 1, 3]

The graph of $y = f'(x)$ is given in the accompanying diagram.



(a) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is zero.

$x = -2$ and 4 ✓✓

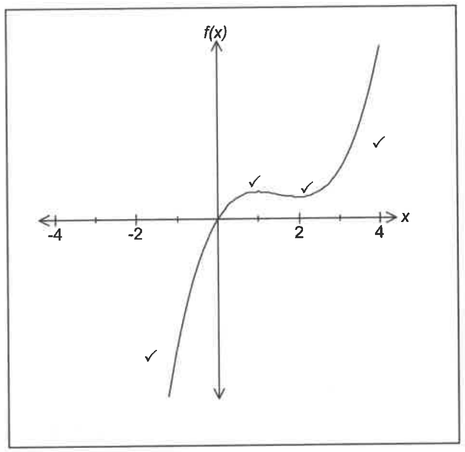
(b) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is a minimum.

$x = 1$ ✓

(c) Sketch on the same axes a possible graph of $y = f(x)$.

5. [4 marks]

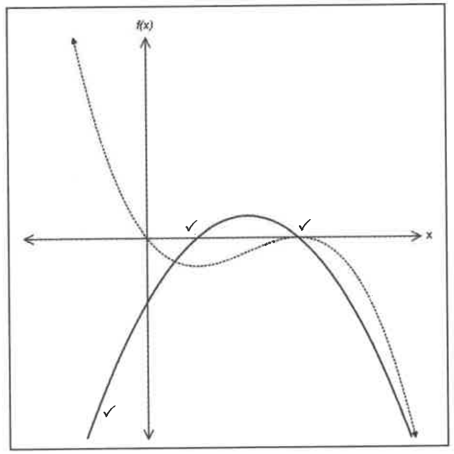
The curve $y = f(x)$ cuts the x-axis at the origin and nowhere else.
$\frac{dy}{dx} = 0$ at $x = 1$ and $x = 2$. $\frac{dy}{dx} < 0$ only for $1 < x < 2$.
Give a possible sketch of $y = f(x)$.



Calculator Free

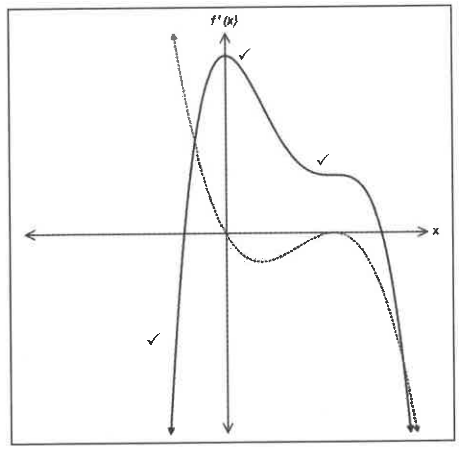
6. [3 marks]

Given the sketch of $y = f(x)$, on the set of axes given, give a possible sketch of $y = f'(x)$.



7. [3 marks]

Given the sketch of $y = f'(x)$, give a possible sketch of $y = f(x)$.



29 Stationary Points & Graphs

Calculator Free

1. [9 marks: 7, 3]

Consider the curve with equation $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{1}{3}$.

(a) Find the coordinates of the stationary point(s) on this curve. Use an appropriate analytical method to determine the nature of these point(s).

$\frac{dy}{dx} = x^2 + x - 2$ ✓

For turning points: $\frac{dy}{dx} = 0 \Rightarrow x^2 + x - 2 = 0$ ✓

$(x-1)(x+2) = 0$

$x = -2, 1$ ✓

For $x = -2$, $y = 3$.

x	-2^-	-2	-2^+
dy/dx	+	0	–

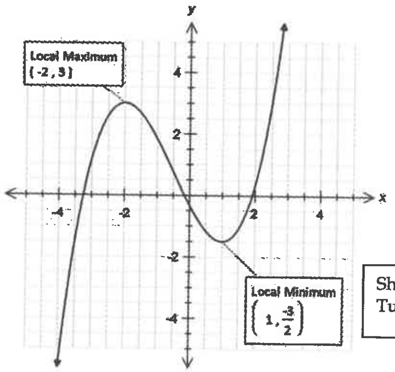
Hence, (–2, 3) is a maximum point. ✓✓

For $x = 1$, $y = -\frac{3}{2}$.

x	1^-	1	1^+
dy/dx	–	0	+

Hence $(1, -\frac{3}{2})$ is a minimum point. ✓✓

(b) Sketch the curve. Indicate clearly the turning points.



Shape	✓
Turning Points	✓✓

Calculator Free

2. [14 marks: 3, 7, 4]

Consider the curve with equation $y = x^3 - 3x + 2$.

(a) Find the roots of this curve.

When $x = 1$, $y = 1 - 3 + 2 = 0$

Hence, $x = 1$ is a root. ✓

Therefore, $y = x^3 - 3x + 2 = (x-1)(x^2 + x - 2)$ ✓

$= (x-1)(x-1)(x+2)$

Roots are: $x = -2$ and $x = 1$. ✓

(b) Use a calculus method to determine the minimum and maximum points on this curve.

$\frac{dy}{dx} = 3x^2 - 3$ ✓

For turning points: $\frac{dy}{dx} = 0 \Rightarrow 3(x^2 - 1) = 0$ ✓

$x = -1, 1$ ✓

For $x = -1$, $y = 4$.

x	-1^-	-1	-1^+
dy/dx	+	0	–

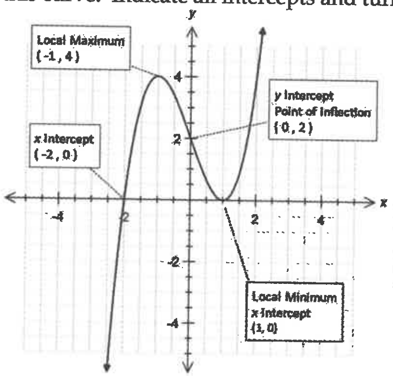
Hence, (–1, 4) is a maximum point. ✓✓

For $x = 1$, $y = 0$.

x	1^-	1	1^+
dy/dx	–	0	+

Hence (1, 0) is a minimum point. ✓✓

(c) Hence, sketch this curve. Indicate all intercepts and turning points.



Intercepts:	✓✓
Turning Points	✓✓

Calculator Free

3. [11 marks: 5, 3, 3]

Consider the curve with equation $y = x^3 - 6x^2 + 12x - 9$.

(a) Find the coordinates of the stationary point(s) on this curve. Use a calculus method to determine the nature of these point(s).

$\frac{dy}{dx} = 3x^2 - 12x + 12$ ✓

For stattionary points: $\frac{dy}{dx} = 0 \Rightarrow 3(x^2 - 4x + 4) = 0$ ✓

$(x-2)^2 = 0$

$x = 2$ ✓

For $x = 2$, $y = -1$.

x	2^-	2	2^+
dy/dx	+	0	+

Hence, (2, –1) is a horizontal inflection point. ✓✓

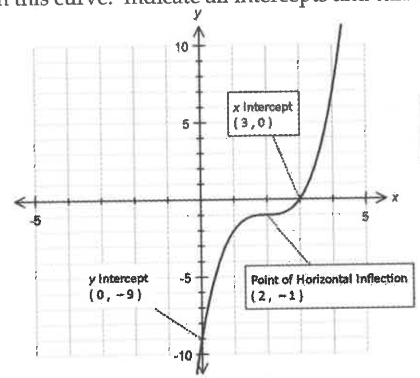
(b) Find the coordinates of all the intercepts.

When $x = 0$, $y = -9$. Hence (0, –9). ✓

Cubic has a horizontal inflection point.
Therefore, there must only be one root.
When $x = 3$, $y = 27 - 18 + 12 - 1 = 0$.
Hence (3, 0). ✓✓

Or (2, –1) is a horizontal inflection point.
Hence, $y = (x-2)^3 - 1$.
When $y = 0$, $x = 3$.
Hence (3, 0).

(c) Hence, sketch this curve. Indicate all intercepts and turning points.



Intercepts: ✓✓
Inflection Point ✓

Calculator Free

4. [5 marks]

The curve $y = 3x^3 + ax^2 + bx + c$ has a y-intercept at (0, 4) and stationary points at $x = 0$ and $x = 2$. Find the values of a, b and c.

$x = 0, y = 4 \Rightarrow c = 4$ ✓

Hence, $y = 3x^3 + ax^2 + bx + 4$.

Gradient function $y' = 9x^2 + 2ax + b$ ✓

When $x = 0, y' = 0. \Rightarrow b = 0$ ✓

When $x = 2, y' = 0. \Rightarrow 9(2)^2 + 2a(2) = 0 \Rightarrow a = -9$ ✓✓

5. [6 marks]

Consider the curve with equation $y = ax^3 + bx^2 - 12x + c$.
The curve has a turning point at (–1, 15) and another turning point at $x = 2$.
Find a, b and c. Show clearly how you obtained your answer.

$\frac{dy}{dx} = 3ax^2 + 2bx - 12$ ✓

When $x = -1$, $\frac{dy}{dx} = 0 \Rightarrow 3a - 2b - 12 = 0$ I ✓

When $x = 2$, $\frac{dy}{dx} = 0 \Rightarrow 12a + 4b - 12 = 0$

$6a + 2b - 6 = 0$ II ✓

Solve I and II simultaneously: $a = 2, b = -3$ ✓✓

Hence, $y = 2x^3 - 3x^2 - 12x + c$.
When $x = -1$, $y = 15$. $-2 - 3 + 12 + c = 15$
$c = 8$ ✓

Calculator Free

6. [10 marks]

A curve has equation $y = ax^3 + bx^2 + cx + d$. The curve has a turning point at $x = 1$, a y-intercept at (0, –33) and a tangent with equation $y = -24x - 37$ at $x = -1$. Find the values of a, b, c and d. Show clearly how you obtained your answer.

y-intercept at (0, –33) $\Rightarrow$ $d = -33$ ✓

$\frac{dy}{dx} = 3ax^2 + 2bx + c$ ✓

Turning Point at $x = 1$ $\Rightarrow$ $3a + 2b + c = 0$ I ✓

Gradient at $x = -1$ is –24 $\Rightarrow$ $3a - 2b + c = -24$ II ✓

I – II $\quad 4b = 24 \Rightarrow b = 6$ ✓

At the point of contact between tangent and curve:
$x = -1$, $y = -24(-1) - 37 = -13$.
Hence, (–1, –13) also lies on this curve. ✓

For the curve: $-a + 6 - c - 33 = -13$

$a + c = -14$ ✓

From I $\quad 3a + c = -12$ ✓

Hence: $\quad a = 1, c = -15.$ ✓✓

Calculator Assumed

7. [8 marks]

A curve has equation $y = x^4 - x^2 - 4$.
Use a calculus method to find the coordinates of all the stationary points on this curve. Use the sign test to determine the nature of each of these points.

$\frac{dy}{dx} = 4x^3 - 2x$ ✓

For stationary points, $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 2x = 0$

$\Rightarrow x = 0 \text{ or } \pm\frac{1}{\sqrt{2}}$. ✓✓

When $x = 0$, $y = -4$.
Using the sign test: ✓

x	0^-	0	0^+
$\frac{dy}{dx}$	+	0	–

Hence, (0, –4) is a maximum point. ✓

When $x = \frac{1}{\sqrt{2}}$, $y = \frac{-17}{4}$.
Using the sign test: ✓

x	$\frac{1}{\sqrt{2}}^-$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}^+$
$\frac{dy}{dx}$	–	0	+

Hence, $(\frac{1}{\sqrt{2}}, \frac{-17}{4})$ is a minimum point. ✓

When $x = \frac{-1}{\sqrt{2}}$, $y = \frac{-17}{4}$.
Using the sign test: ✓

x	$\frac{-1}{\sqrt{2}}^-$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}^+$
$\frac{dy}{dx}$	–	0	+

Hence, $(\frac{-1}{\sqrt{2}}, \frac{-17}{4})$ is a minimum point. ✓

Calculator Assumed

8. [10 marks: 2, 5, 3]

A curve has equation $y = (x-2)(x^2+1)$.

(a) Find the coordinates of the horizontal and vertical intercepts.

(0, −2) and (2, 0). ✓✓

(b) Use a calculus method to find the exact coordinates of the turning points. Use the sign test to determine the nature of these points.

Use CAS Calculator: $\frac{dy}{dx} = 3x^2 - 4x + 1$ ✓

For stationary points, $\frac{dy}{dx} = 0 \Rightarrow x = 1$ or $\frac{1}{3}$. ✓✓

When $x = 1$, $y = -2$.
Using the sign test:

x	1^-	1	1^+
$\frac{dy}{dx}$	−	0	+

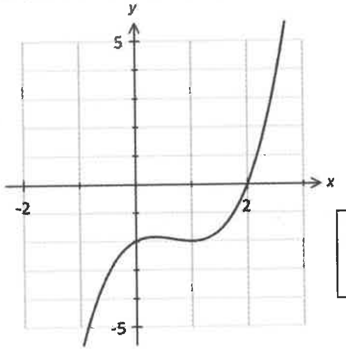
Hence, (1, −2) is a minimum point. ✓

When $x = \frac{1}{3}$, $y = \frac{-50}{27}$.
Using the sign test:

x	$\frac{1}{3}^-$	$\frac{1}{3}$	$\frac{1}{3}^+$
$\frac{dy}{dx}$	+	0	−

Hence, $(\frac{1}{3}, \frac{-50}{27})$ is a maximum point. ✓

(c) Sketch the graph of this curve in the axes below.



Shape ✓
Intercepts ✓
Turning Points ✓

Calculator Assumed

9. [10 marks: 2, 5, 3]

Consider the curve with equation $y = 2x^3 - 9x^2 + 12x - 4$.

(a) State the roots of this curve.

$x = \frac{1}{2}, 2$ ✓✓

(b) Use a calculus method to determine the turning points on this curve. Use an appropriate method to determine the nature of each of these points.

$\frac{dy}{dx} = 6x^2 - 18x + 12$ ✓

For turning points $\frac{dy}{dx} = 0$:

$6x^2 - 18x + 12 = 0$ ✓

$x = 1, 2$ ✓

For $x = 1$, $y = 1$

x	1^-	1	1^+
dy/dx	+	0	−

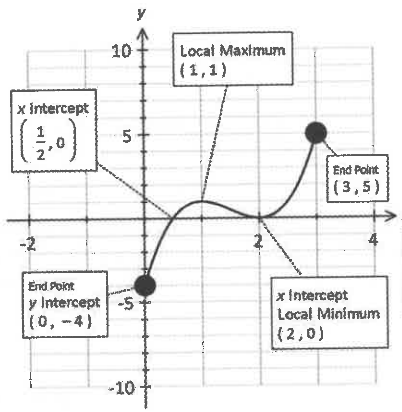
Hence, (1, 1) is a maximum point. ✓

For $x = 2$, $y = 0$

x	0^-	0	0^+
dy/dx	−	0	+

Hence, (2, 0) is a minimum point. ✓

(c) Sketch this curve for $0 \le x \le 3$ in the axes provided below. Label all intercepts and turning points.



End points ✓
Intercepts ✓
Turning Pts ✓

30 Rates of Change

Calculator Assumed

1. [8 marks: 1, 2, 2, 3]

The height of a ball t seconds after it is thrown vertically upwards from ground level is given by $h = 20t - 5t^2$ metres.

(a) Find the height of the ball after 2 seconds.

When $t = 2$, $h = 20$ m ✓

(b) At what rate is the height of the ball changing when $t = 1$ second.

$\frac{dh}{dt} = 20 - 10t$ ✓

When $t = 1$, $\frac{dh}{dt} = 10\text{ ms}^{-1}$ ✓

(c) Find when the rate of change of the height is zero.

$\frac{dh}{dt} = 0 \Rightarrow 20 - 10t = 0.$ ✓

Hence, $t = 2$ seconds ✓

(d) Find the height of the ball when the rate of change of height is -10 ms^{-1}.

$\frac{dh}{dt} = 10 \Rightarrow 20 - 10t = -10.$ ✓

Hence, $t = 3$ seconds ✓

Therefore, $h = 60 - 45 = 15$ m ✓

Calculator Assumed

2. [10 marks: 2, 1, 2, 1, 3, 1]

The volume (V cm^3) of a spherical balloon is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the balloon. The radius of the balloon changes with time t (seconds) according to the rule $r = 10 - t$.

(a) For what values of t is the rule $r = 10 - t$ valid? Why?

Since, t represents time, $t \geq 0$. ✓
For $t > 10$, $r < 0$.
Hence, rule is valid for $0 \leq t \leq 10$. ✓

(b) Find V in terms of t.

Subst. $r = 10 - t$ into V.
$V = \frac{4}{3}\pi(10 - t)^3\text{ cm}^3$ ✓

(c) Find an expression for the rate with which the radius changes with time.

Use CAS calculator: $\frac{dV}{dt} = -4\pi(10 - t)^2\text{ cm}^3\text{s}^{-1}$. ✓✓

(d) Find the rate at which the volume is changing when $t = 5$ seconds. An exact answer is required.

When $t = 5$;
$\frac{dV}{dt} = -4\pi(10 - 5)^2 = -100\pi\text{ cm}^3\text{s}^{-1}$. ✓

(e) Find the exact value of t when the rate at which the volume changes is $-\pi\text{ cm}^3\text{s}^{-1}$.

When $\frac{dV}{dt} = -\pi$; $-4\pi(10 - t)^2 = -\pi$ ✓

$(10 - t)^2 = \frac{1}{4} \Rightarrow t = \frac{19}{2}$ or $\frac{21}{2}$ ✓

Reject $t = \frac{21}{2}$ as $0 \leq t \leq 10$. Hence, $t = \frac{19}{2}$ seconds. ✓

(f) Hence, find the exact volume of the balloon when the rate at which the volume changes is $-\pi\text{ cm}^3\text{s}^{-1}$.

$V = \frac{4}{3}\pi(10 - \frac{19}{2})^3$

$= \frac{\pi}{6}\text{ cm}^3$ ✓

Calculator Assumed

3. [10 marks: 1, 2, 1, 1, 2, 3]

The mass (M g) of a crystal being grown in a laboratory at time t hours is given by $M = -\frac{1}{30}t^3 - \frac{1}{20}t^2 + 50t + 5$ for $0 \le t \le 20$.

(a) Find the change in mass of the crystal between $t = 0$ and $t = 10$ hours.

> Change in mass = M(10) – M(0)
> $= 466\frac{2}{3} - 5 = 461\frac{2}{3}$ g. ✓

(b) Find the average rate of change of mass of the crystal in the first 10 hours.

> Average rate of change $= \frac{M(10) - M(0)}{10 - 0}$ ✓
> $= \frac{461\frac{2}{3}}{10} = 46.17$ g/hr ✓

(c) Find an expression for the instantaneous rate of change of mass of the crystal with respect to time.

> $\frac{dM}{dt} = -\frac{1}{10}t^2 - \frac{1}{10}t + 50$ ✓

(d) Find the instantaneous rate of change of mass of the crystal at $t = 10$ hours.

> When $t = 10$;
> $\frac{dM}{dt} = -\frac{1}{10} \times 10^2 - \frac{1}{10} \times 10 + 50 = 39$ g/hr ✓

(e) Comment on the difference between your answers in part (b) and (d).

> Answer in (d) is the rate of change at that instant in time. ✓
> Answer in (b) refers to the average rate of change within an interval of time. ✓

(f) Find the mass of the crystal when the instantaneous rate of change of mass is 48 g per hour.

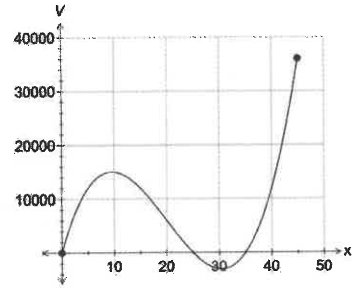
> $-\frac{1}{10}t^2 - \frac{1}{10}t + 50 = 48$ ✓
> $t^2 + t - 20 = 0 \Rightarrow t = 4$ ✓
> Reject $t = -5$ as $t \ge 0$.
> M(4) = 202.07 g ✓

31 Optimisation

Calculator Assumed

1. [4 marks: 2, 2]

The volume of a box of height x cm is given by $V = x(50 - 2x)(70 - 2x)$ cm^3. The graph of V against x is drawn below for $0 \le x \le 45$ cm.



(a) Give the possible values of x.

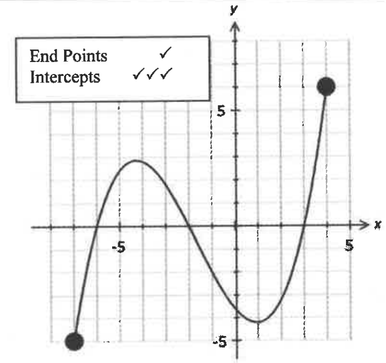
> $0 < x < 25$ ✓✓

(b) Use the graph to estimate the maximum possible volume of the box and give the associated value of x.

> Max Volume ≈ 15 000 cm^3. ✓
> When $x \approx 9.5$ cm ✓

2. [6 marks: 4, 1, 1]

In the axes provided, sketch the graph of $f(x) = 0.1x^3 + 0.5x^2 - 1.2x - 3.6$ for $-7 \le x \le 4$. Indicate clearly the main features of the curve.



(a) Find to 2 decimal places the maximum value for $f(x)$ for $-7 \le x \le 0$.

> $f(-4.27) = 2.86$ ✓

(b) Find to 2 decimal places the minimum value for $f(x)$ for $-7 \le x \le 4$.

> $f(-7) = -5.00$ ✓

Calculator Assumed

3. [11 marks: 5, 3, 3]

Consider the curve with equation $y = (x-5)(x+1)^2 = x^3 - 3x^2 - 9x - 5$.

(a) Use a calculus method to determine the turning points on this curve. Use an appropriate method to determine the nature of each of these points.

$\frac{dy}{dx} = 3x^2 - 6x - 9$ ✓

For turning points $\frac{dy}{dx} = 0$: $3x^2 - 6x - 9 = 0$ ✓

$x = -1, 3$ ✓

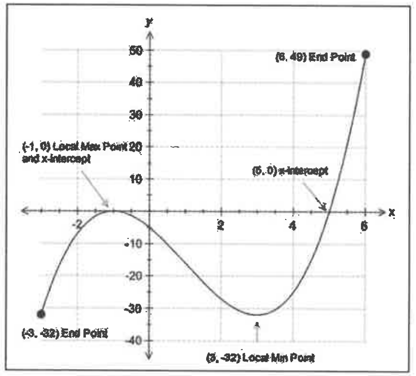
For $x = -1, y = 0$
Hence, $(-1, 0)$ is a maximum point. ✓

x	1^-	1	1^+
dy/dx	+	0	–

For $x = 3, y = -32$
Hence, $(3, -32)$ is a minimum point. ✓

x	0^-	0	0^+
dy/dx	–	0	+

(b) Sketch this curve for $-3 \le x \le 6$ in the axes provided. Label all intercepts and turning points.



End points ✓
Intercepts ✓
Turning Pts ✓

(c) For $y = (x-5)(x+1)^2 = x^3 - 3x^2 - 9x - 5$ in the domain $-3 \le x \le 6$, find:
(i) the largest value of y and the corresponding x-value

Global max for $y = 49$ when $x = 6$ ✓

(ii) the lowest value for y and the corresponding x-value.

Global min for $y = -32$ when $x = -3$ and 3 ✓✓

Calculator Assumed

4. [8 marks: 1, 5, 2]

The number of litres, V (kL), of unleaded petrol, sold at an outlet each day, is modelled by $V = \frac{t^3}{3} - 9t^2 + 65t + 50$, for $0 \le t \le 14$, where t is the number of days in the given fortnight. Use an analytical method to find:

(a) how many litres of unleaded petrol was sold at the start of the fortnight

When $t = 0$, V = 50 kL ✓

(b) the minimum amount of unleaded petrol sold per day and the corresponding t value

$V' = t^2 - 18t + 65$ ✓
When $V' = 0$, $t = 5, 13$. ✓
V(5) = 191.67 kL
V(13) = 106.33 kL ✓

End points: V(0) = 50 and V(14) = 110.7 kL. ✓

Hence, minimum amount = 50 kL when $t = 0$. ✓

(c) the maximum amount of unleaded petrol sold per day and the corresponding t value.

Hence, maximum amount = 191.67 kL when $t = 5$. ✓✓

5. [8 marks: 1, 1, 6]

The organisers of a charity ball believe that if the ball tickets are priced at $80 each they would be able to sell 500 tickets. For each $5 increase in the price of each ticket, they expect the sales to decrease by 10 tickets.

(a) Find the number of tickets they expect to sell if the price of each ticket is increased by x lots of $5.

$(500 - 10x)$ tickets. ✓

(b) Find the expected revenue when the price of each ticket is raised by x lots of $5.

Revenue $R = (500 - 10x)(80 + 5x)$ ✓

Calculator Assumed

5. (c) Use Calculus to find the price per ticket that will maximize the revenue of the organisers. Give the maximum revenue.

Use CAS calculator: R' = 1 700 – 100x ✓
When R' = 0, x = 17. ✓
Using the sign test: ✓

x	17^-	17	17^+
$\frac{dy}{dx}$	+	0	–

Hence R is maximum when x = 17. ✓

Hence, price per ticket = 80 + 5 × 17 = $165. ✓
Maximum Revenue = $54 450. ✓

---

6. [8 marks: 6, 2]

The population of dingoes in a large nature reserve is modelled by $P = t^3 - 35t^2 + 275t + 875$ for $0 \le t \le 25$, where t is time in years after Jan 2000.

(a) Use Calculus to find the population at its lowest level. Give the year when this occurred.

$P'(t) = 3t^2 - 70t + 275$ ✓
$P'(t) = 0 \Rightarrow t = 5, \frac{55}{3}$. ✓
For $t = \frac{55}{3}$, P = 314.8148148.
Hence, P has a local minimum at $t = \frac{55}{3}$. ✓✓

t	$\frac{55}{3}^-$	$\frac{55}{3}$	$\frac{55}{3}^+$
dy/dx	–	0	+

End points: P(0) = 875, P(25) = 1500 ✓
Hence min. population is 315 in 2018. ✓

(b) Use Calculus to find the population at its highest level between 2000 and 2025 inclusive. Give the year when this occurred.

P(5) = 1500
End points: P(0) = 875, P(25) = 1500 ✓

Hence, max. population is 1500 in 2005 and 2025. ✓

Calculator Assumed

7. [8 marks: 3, 6]

A rectangular sheet of cardboard, 10 cm by 15 cm, is to be made into an open rectangular box. Four squares, each of side, x cm, are removed from each corner of the cardboard to form the net of the box.

(a) Show that the volume, V, of the box is given by $V = x(15 - 2x)(10 - 2x)$.

Height = x cm
Hence, width = 10 – 2x cm and length = 15 – 2x cm. ✓✓
Therefore, V = x(15 – 2x)(10 – 2x). ✓

(b) Use a calculus method to find the dimensions of the box that will maximise its volume.

Use CAS calculator: $V' = 12x^2 - 100x + 150$ ✓
When V' = 0, x = 1.96, (reject x = 6.37 as $0 < x < 5$. ✓

V(1.96) = 132.04 ✓

Hence, V is maximum when
height = 1.96 cm, length = 11.08 cm and width = 6.08 cm. ✓✓✓

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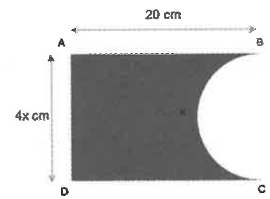
8. [6 marks: 2, 4]

The figure shown in the diagram is obtained by removing a semi-circle BKC from the rectangle ABCD. AB = 20 cm and AD = 4x cm. The perimeter of the figure ABKCD is 200 cm.



(a) Show that the area of figure ABKCD is given by $A = 80x - 2\pi x^2$ cm^2.

Area = Area of rectangle ABCD – Area of semi-circle BKC ✓
$= 20 \times 4x - \frac{1}{2} \times \pi \times (2x)^2$ ✓
$= 80x - 2\pi x^2$.

Calculator Assumed

8. (b) Use calculus techniques to find in terms of π, the value of x that will maximise A. State this maximum value, in terms of π.

$\frac{dA}{dx} = 80 - 4\pi x$ ✓

For Max/Min, $\frac{dA}{dx} = 0. \Rightarrow 80 - 4\pi x = 0$ ✓

$x = \frac{20}{\pi}$ ✓

$\Rightarrow A = \frac{800}{\pi}$ ✓

9. [8 marks]

A closed rectangular box, has a volume of 10 000 cm³. The height of the box is twice its width. Use a calculus method to find the dimensions of the box that will minimise its surface area.

Let width = x cm, height = $2x$ cm and length = y cm. ✓

Volume $V = 2x^2y = 10\,000$.

$\Rightarrow y = \frac{5\,000}{x^2}$. ✓

Hence, surface area $S = 2xy + 2(2x^2) + 2(2xy)$

$= 6xy + 4x^2$

$= 4x^2 + 6x \times \frac{5\,000}{x^2}$

$= 4x^2 + \frac{30\,000}{x}$. ✓✓

Use CAS calculator: $S' = 8x - \frac{30\,000}{x^2}$. ✓

When $S' = 0$, $x = 15.5362$.

Using the sign test: ✓

x	15.54^-	15.54	15.54^+
$\frac{dy}{dx}$	–	0	+

Hence S is minimum when $x = 15.5362$ cm. ✓

Dimensions of box: height = 31.07 cm, width = 15.54 cm, length = 20.71 cm ✓

Calculator Assumed

10. [10 marks: 4, 6]

The total surface area of a closed rectangular box is 2 000 cm². The length of the box is four times its height x cm.

(a) Show that the volume of the box is given by $V = 800x - 3.2x^3$

Let width of box be w.

Surface area $= 2(4x \times w) + 2(w \times x) + 2(4x \times x)$ ✓

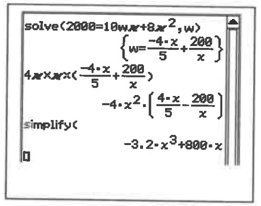
$= 10wx + 8x^2$

Hence $2000 = 10wx + 8x^2$ ✓

$w = \frac{2000 - 8x^2}{10x}$ ✓

Hence Volume $V = 4x \times x \times \frac{2000-8x^2}{10x}$ ✓

$= 800x - 3.2x^3$



(b) Use a calculus method to find the maximum volume of the box and the corresponding dimensions of the box.

$\frac{dV}{dx} = -9.6x^2 + 800$ ✓

$\frac{dV}{dx} = 0 \Rightarrow x = -9.1287$ or $x = 9.1287$ ✓

Clearly $x > 0$.

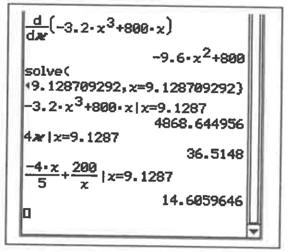
Hence V is maximised when $x = 9.1287$

$V_{max} = 4868.64$ cm³ ✓

Height of box ≈ 9.1 cm ✓

Length of box ≈ 36.5 cm ✓

Width of box ≈ 14.6 cm ✓



32 Anti-Differentiation

Calculator Free

1. [9 marks: 1, 1, 1, 2, 2, 2]

Find the anti-derivative of each of the following:

(a) $3x^2 + 4$

> Anti-derivative = $x^3 + 4x + C$ ✓

(b) $\frac{x^3}{2}$

> Anti-derivative = $\frac{x^4}{8} + C$ ✓

(c) $\frac{4x^3}{5}$

> Anti-derivative = $\frac{x^4}{5} + C$ ✓

(d) $\frac{3x^2 + 5x^3}{4}$

> $\frac{3x^2+5x^3}{4} = \frac{3x^2}{4} + \frac{5x^3}{4}$ ✓
>
> Anti-derivative = $\frac{x^3}{4} + \frac{5x^4}{16} + C$ ✓

(e) $(x^2 + 1)^2$

> $(x^2+1)^2 = x^4 + 2x^2 + 1$ ✓
>
> Anti-derivative = $\frac{x^5}{5} + \frac{2x^3}{3} + x + C$ ✓

(f) $\frac{-2x^5 + 5x^4}{3x^2}$

> $\frac{-2x^5 + 5x^4}{3x^2} = \frac{-2x^3}{3} + \frac{5x^2}{3}$ ✓
>
> Anti-derivative = $\frac{-x^4}{6} + \frac{5x^3}{9} + C$ ✓

Calculator Free

2. [3 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = 2x + 5$. Find the equation of the curve if it passes through (−1, 3).

> $y = x^2 + 5x + C$ ✓
>
> When $x = -1, y = 3, \Rightarrow C = 7$. ✓
>
> Hence, $y = x^2 + 5x + 7$ ✓

3. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = -\frac{4x^3}{3} + 2x - 1$. Find the equation of the curve if it passes through (1, 2).

> $y = \frac{-x^4}{3} + x^2 - x + C$ ✓✓
>
> When $x = 1, y = 2, \Rightarrow C = \frac{7}{3}$. ✓
>
> Hence, $y = \frac{-x^4}{3} + x^2 - x + \frac{7}{3}$ ✓

4. [4 marks]

Find $f(x)$ if $f'(x) = x^2 + 2x + k$ and $f(0) = -2$ and $f(-1) = \frac{-1}{3}$.

> $f(x) = \frac{x^3}{3} + x^2 + kx + C$ ✓
>
> $f(0) = -2 \Rightarrow C = -2$ ✓
>
> $f(-1) = \frac{-1}{3} \Rightarrow k = -1$ ✓
>
> Hence, $f(x) = \frac{x^3}{3} + x^2 - x - 2$ ✓

33 Rectilinear Motion

Calculator Assumed

1. [11 marks: 1, 2, 2, 3, 3]

A particle P moves along a straight line. Its displacement s metres, t seconds after passing a fixed point O is given by $s = -3t^2 + 4t + k$.

(a) Find the value of k.

When $t = 0, s = 0. \Rightarrow k = 0$ ✓

(b) Find an expression for the velocity t seconds after passing O. Hence, find the velocity of P as it passed O the first time.

Velocity $v = \frac{ds}{dt} = -6t + 4$ ✓
When $t = 0, v = 4 \text{ ms}^{-1}$ ✓

(c) Find when P is travelling with a velocity of 2 ms^{-1}.

When $v = 2$, $-6t + 4 = 2$ ✓
$\Rightarrow t = \frac{1}{3}$ seconds ✓

(d) Find when P is travelling with a speed of 2 ms^{-1}.

When speed = 2, $|-6t + 2| = 2$. ✓
$\Rightarrow$ $-6t + 4 = 2$ or $-6t + 4 = -2$
$\Rightarrow t = \frac{1}{3}$ or $t = 1$ ✓✓

(e) Find when P is 1 metre away from O.

When P is 1 m away, $|s| = 1$.
Hence, $-3t^2 + 4t = 1$ or $-3t^2 + 4t = -1$ ✓
$\Rightarrow t = \frac{1}{3}, 1$ or $t = 1.55$ (reject -0.22)
Hence, P is 1 m away from O when $t = \frac{1}{3}$, 1 or 1.55 seconds. ✓✓

Calculator Assumed

2. [10 marks: 1, 2, 2, 1, 2, 1, 1]

The displacement of a particle moving along a straight line at time t seconds is given by $s = t^3 - \frac{9}{2}t^2 + 6t$ metres.

(a) Find the displacement of the particle at time $t = 1$ seconds.

$s(1) = \frac{5}{2}$ m ✓

(b) Find the change in displacement in the first 2 seconds.

Change in displacement $= s(2) - s(0)$ ✓
$= 2 - 0 = 2$ m ✓

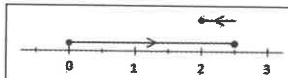
(c) Find the velocity of the particle at $t = 2$ seconds.

Velocity $v = \frac{ds}{dt} = 3t^2 - 9t + 6$ ✓
$v(2) = 0 \text{ ms}^{-1}$. ✓

(d) Find when the particle changes direction.

$v = 0 \Rightarrow 3t^2 - 9t + 6 = 0$
$t = 1, 2$ seconds ✓

(e) Find the distance travelled in the first two seconds.



$s(0) = 0$ $s(1) = \frac{5}{2}$ $s(2) = 2$

Hence, distance travelled $= \frac{5}{2} + \frac{1}{2} = 3$ m ✓✓

(f) Find the average speed in the first two seconds.

Average speed $= \frac{3}{2} \text{ ms}^{-1}$. ✓

(g) What does the difference between your answers in (b) and (e) imply?

The particle experienced at least one change of direction within the first 2 seconds. ✓

Calculator Assumed

3. [7 marks: 1, 3, 1, 2]

The displacement of a body at time t seconds is given by $s = 4t + \dfrac{1}{1+t}$ metres.

(a) Find an expression for the velocity of the body at time t seconds.

Velocity $v = \dfrac{ds}{dt} = 4 - \dfrac{1}{(1+t)^2}$ ✓

(b) Show that the body is never stationary.

For the body to be stationary,

$4 - \dfrac{1}{(1+t)^2} = 0$ ✓

$t = -\frac{1}{2}$ or $-\frac{3}{2}$ ✓

But time $t \geq 0$. ✓

Hence, the body is never stationary.

(c) Find an expression for the acceleration at time t seconds.

Acceleration $a = \dfrac{dv}{dt} = \dfrac{2}{(1+t)^3}$ ✓

(d) Hence, describe the motion of the body for large values of t.

$v = 4 - \dfrac{1}{(1+t)^2}$.

For large values of t, $\dfrac{1}{(1+t)^2} \to 0$. ✓

Hence, $v \to 4$.

That is, for large values of t the body moves with a constant velocity of 4 ms^{-1}. ✓

Calculator Assumed

4. [10 marks: 4, 2, 4]

The displacement of a body moving along a straight line is given by $s = -t^3 + at^2 + bt + 3$ metres where t is time in seconds. The initial velocity of the body is 5 ms^{-1}. The body is momentarily at rest when $t = 1$ second.

(a) Find the values of a and b.

Velocity $v = \dfrac{ds}{dt} = -3t^2 + 2at + b$ ✓

When $t = 0$, $v = 5$. $\Rightarrow b = 5$ ✓

Hence, $v = -3t^2 + 2at + 5$

Body is at rest when $t = 1$. $\Rightarrow -3 + 2a + 5 = 0$ ✓

Hence, $a = -1$ ✓

(b) Find when the body changes direction.

When body changes direction,

$v = 0$;

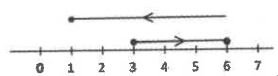
$\Rightarrow -3t^2 - 2t + 5 = 0$ ✓

$t = 1$ second (reject $\frac{-5}{3}$) ✓

(c) Find the instantaneous speed at $t = 2$ seconds and the average speed in the first 2 seconds.

Instantaneous speed at $t = 2$,

$v(2) = -11$ ms^{-1}. ✓



$s(0) = 3$ $s(1) = 6$ $s(2) = 1$

Distance travelled in the first 2 seconds

$= 3 + 5 = 8$ m ✓✓

Hence, average speed in the first 2 seconds

$= 4$ ms^{-1}. ✓

Calculator Assumed

5. [6 marks: 2, 2, 2] [TISC]

The velocity v cms^{-1}, of a particle P moving in a straight line at a point x cm from the origin is given by the equation $v^2 = -\int x\,dx$. P starts from the origin with a velocity of 10 cms^{-1}.

(a) Show that $v^2 = 100 - \frac{x^2}{2}$.

$v^2 = -\frac{x^2}{2} + C$ ✓

When $x = 0$, $v = 10$:

$C = 100$ ✓

Hence, $v^2 = 100 - \frac{x^2}{2}$

(b) Find where P is instantaneously at rest.

$v = 0 \Rightarrow 100 - \frac{x^2}{2} = 0$

$x = \pm 10\sqrt{2}$ cm ✓✓

(c) Find the maximum speed of P and state where it occurs.

When $x = 0$, $v^2 = 100$

$v = \pm 10$ cms^{-1}.

Hence, max $v = 10$ cms^{-1} at $x = 0$. ✓✓

Calculator Assumed

6. [5 marks: 2, 3]

The acceleration (ms^{-2}) of a particle moving along a straight line is given by $a = 4t + 1$, where t is time in seconds. At $t = 1$, the velocity of the particle is 5 ms^{-1} and the displacement of the particle is 10 m.

(a) Find an expression for the velocity of the particle at any time t.

$v = \int 4t + 1\,dt$

$= 2t^2 + t + C$ ✓

When $t = 1, v = 5 \Rightarrow C = 2$

Hence, $v = 2t^2 + t + 2$. ✓

(b) Find an expression for the displacement of the particle at any time t.

Displacement $x = \int 2t^2 + t + 2\,dt$

$= \frac{2t^3}{3} + \frac{t^2}{2} + 2t + K$ ✓

When $t = 1, x = 10. \Rightarrow K = \frac{41}{6}$ ✓

Hence, $x = \frac{2t^3}{3} + \frac{t^2}{2} + 2t + \frac{41}{6}$. ✓

Calculator Assumed

7. [10 marks: 1, 6, 3]

A particle starts off from a fixed point 0 with an acceleration (mms^{-2}) of $a = mt - 24$, where t is time in seconds. The particle travels in a straight line and returns to O at $t = 4$ seconds and has a change of displacement of –9 mm in the third second (it moves in the same direction during this time).

(a) Find in terms of m an expression for the velocity of the particle at any time t.

Velocity $v = \int mt - 24\ dt$

$= \frac{mt^2}{2} - 24t + C$ ✓

(b) Find the displacement of the particle at any time t.

Displacement $x = \frac{mt^3}{6} - 12t^2 + Ct + K$ ✓

When $t = 0, x = 0$. $\Rightarrow K = 0$.

Hence, $x = \frac{mt^3}{6} - 12t^2 + Ct$ ✓

When $t = 4, x = 0$. $\Rightarrow \frac{64m}{6} + 4C = 192$ (I) ✓

Also, $x(3) - x(2) = -9$. $\Rightarrow \frac{19m}{6} + C = 51$ (II) ✓

Solve I and II simultaneously, $m = 6, C = 32$

Hence, $x = t^3 - 12t^2 + 32t$ ✓✓

(c) Find when the particle is at O the third time (if it does).

When it is at O, $x = 0$.

$\Rightarrow t^3 - 12t^2 + 32t = 0$ ✓

$t = 0, 4, 8$ ✓

Hence, the particle is at O for the third time at $t = 8$ seconds. ✓